On the Correlated $K$-Distribution With Arbitrary Fading Parameters

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Abstract—The correlated bivariate $K$-distribution with arbitrary and not necessarily identical parameters is introduced and analyzed. Novel infinite series expressions for the joint probability density function and moments are derived for the general case where the associated bivariate distributions, i.e., Rayleigh and gamma, are both arbitrary correlated. These expressions generalize previously known analytical results obtained for identical parameter cases. Furthermore, considering independent gamma distributions, the cumulative distribution and characteristic functions are analytically obtained. Although the derived expressions can be used in a wide range of applications, this letter focuses on the performance analysis of dual branch diversity receivers. Specifically, the outage performance of dual selection diversity receivers operating over correlated $K$ fading/shadowing channels is analytically evaluated. Moreover, for low normalized outage threshold values, closed-form expressions are obtained.

Index Terms—Bivariate statistics, composite fading/shadowing channels, correlated $K$-distribution, outage probability, radar clutter modeling.

I. INTRODUCTION

The composite $K$-distribution has been used in the past to model electromagnetic scattering from physical media, such as tropospheric propagation of radio waves, optical scintillation from the atmosphere, and various types of sea and land radar clutters [1], [2]. In recent years, this generic distribution has been also applied in the scientific field of digital communications over fading/shadowing channels [3]–[5]. The main reason for this choice is that the $K$-distribution can model composite fading environments where multipath (short-term) fading is superimposed on shadowing (long-term fading), e.g., land mobile satellite links [3, Ch. (2.2.3)], [6]. Such channel models exist in applications where signal envelopes follow the Rayleigh distribution and their average powers follow the gamma distribution. In all these applications of the $K$-distribution, it is important to further consider the effects of correlation present in various real-life practical scenarios.

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In contrast to others bivariate (correlated) distributions, e.g., Rayleigh, Nakagami-$m$, Weibull, and generalized-gamma [3], [7]–[9], the correlated $K$-distribution has not been thoroughly studied in the open research literature. For example, previous studies have considered only identical shaping and scaling parameters for the correlated $K$-distribution [10]–[12]. In [11], the joint probability density function (PDF) and the moments of two clutter amplitudes following the $K$-distribution have been derived. Extending this work, also under the assumption of identical parameters, [12] has considered correlation of the powers of the clutter amplitudes.

Motivated by the preceding, in this letter, we generalize related previous research activities by presenting novel analytical expressions for the bivariate $K$-distribution, with arbitrary and not necessarily identical parameters. Moreover, the proposed analysis is general enough to consider correlation in both the signal envelopes and their average powers. These expressions include the joint PDF and moments, while the cumulative distribution function (CDF) and the characteristic function (CF) are obtained for the case of uncorrelated powers. Capitalizing on the derived CDF, the outage probability (OP) at the output of dual-branch selection diversity (SD) receivers operating over correlated $K$ fading channels is derived and its performance is evaluated.

II. CORRELATED $K$-DISTRIBUTION

In the first part of this section, the joint PDF and moments of the $K$-distribution are presented for the generic case where the constituted distributions have different correlation coefficients. In the second part, by considering independent gamma distributions, analytical expressions for important statistical characteristics of the $K$-distribution are obtained.

A. General Case

The $K$-distribution is a mixture of zero mean complex Gaussian and gamma random variables (RVs) [3]. Let $R_i$ ($i = 1$ and 2) be the envelopes of two zero mean complex Gaussian RVs and $G_i$ their mean square envelope values, i.e., $G_i = E_{R_i}(x_i^2)$, with $E_{R_i}(\cdot)$ denoting averaging over the distribution of $R_i$. Then, $R_i$ are Rayleigh RVs with conditional, on $G_i$’s, bivariate PDF given by

$$f_{R_1|R_2}[G_1,R_2|x_1,x_2,G_2] = \sum_{i=0}^{\infty} \frac{A_i}{(1-\rho R_i)G_i^{2r+1}}$$

$$\times \prod_{i=1}^{2} \frac{G_i^{2r+1}}{G_i^{2r+1}} \exp \left[ -\frac{x_i^2}{(1-\rho R_i)G_i} \right]$$

(1)
with $\rho_G$ being the power correlation coefficient between $R_i^2$ and $R_j^2$. Considering $G_1$ and $G_2$ to be correlated gamma RVs, their joint PDF can be derived using [7, eq. (12)] as

$$f_{G_1,G_2}(x_1, x_2) = \sum_{h=0}^{\infty} \frac{(k_1)_h \rho_G^h}{h!(1-\rho_G)^h} \left[ k_2-k_1; k_2+k_1 + h \right]_1 F_1 \left[ k_2-k_1; k_2+k_1 + h; \frac{\rho_G^{x_2}}{(1-\rho_G)^h} \right] \Gamma(k_2 + h) \Omega_i^{h+k+1} \times \prod_{i=1}^{2} \left[ x_i^{h+1} \exp \left[ -x_i \left( \frac{(1-\rho_G)^h}{\Gamma(k_2 + h)} \right) \right] \right]$$

(2)

where $k_i > 0$ is the shaping parameter, $\Omega_i = E_x(\langle x^2_i \rangle)$, $\rho_G$ is the correlation coefficient between $G_1$ and $G_2$, $F_1(c_1, c_2, \cdots)$ is the confluent hypergeometric function [13, eq. (9.210/1)], $(\alpha)_n$ is the Pochhammer symbol [14, eq. (6.1.22)], and $\Gamma(\cdot)$ is the gamma function [13, eq. (8.310/1)]. Without loss of generality, it is assumed throughout this letter that $k_2 \geq k_1$ [7].

The correlated $K$-distribution can be obtained by averaging the conditional on $G_1$’s (1), over the bivariate gamma PDF (2), i.e.,

$$f_{X_1,X_2}(x_1, x_2) = \int_{0}^{\infty} f_{R_1,G_1} f_{G_1,G_2}(x_1, y, x_2) f_{G_2}(v, w) \, dv \, dw \, dx_2$$

(3)

Substituting (1) and (2) in (3), an integral of the form $I = \int_{0}^{\infty} y^n \exp \left[ -A_1 y^{n-1} - A_2 y \right] F_1(1; B; \rho_R A_2 y) dy$ needs to be solved. This integral can be solved by using the infinite series representation of $F_1(c_1, c_2, \cdots)$ [13, eq. (9.210/1)] and then applying [13, eq. (3.471/9)]. Hence, in the general case with different correlation coefficients and not necessarily identical statistical parameters, the joint PDF of the $K$-distribution can be obtained as

$$f_{X_1,X_2}(x_1, x_2) = \sum_{h=0}^{\infty} \frac{16(k_1)_h \rho_G^h}{h! \Gamma(k_2+k_1+1)} \frac{k_2-k_1}{(1-\rho_G)^{k_2+1}} \times \frac{1}{(1-\rho_G)^{k_2+1}} \prod_{i=1}^{2} \left[ x_i^{h+1} \exp \left[ -x_i \left( \frac{(1-\rho_G)^h}{\Gamma(k_2 + h)} \right) \right] \right]$$

(4)

where $K_i(\cdot)$ is the second kind of modified Bessel function of order $\tau_i$ [13, eq. (8.407/1)] and $\tau_i = k_2+h+i(1-i)$. For $\Omega_1 = \Omega_2$ and $k_1 = k_2$, (4) simplifies to a previously known expression [12, eq. (29)], i.e., the PDF of the correlated $K$-distribution with identical parameters.

Using (4) and [13, eq. (6.561/16) and (9.14/1)], the joint moments $\mu(n_1, n_2) = E_x(\langle x_1^{n_1} x_2^{n_2} \rangle)$ can be expressed as

$$\mu_{X_1,X_2}(n_1, n_2) = \left[ \prod_{i=1}^{2} (1-\rho_G)^{k_2+k_1+1} \right] \frac{1}{(1-\rho_G)^{k_2+1}} \right] \times \frac{1}{(1-\rho_G)^{k_2+1}} \prod_{i=1}^{2} \left[ x_i^{h+1} \exp \left[ -x_i \left( \frac{(1-\rho_G)^h}{\Gamma(k_2 + h)} \right) \right] \right]$$

(5)

B. Independent Gamma RVs Case

For $\rho_G = 0$, i.e., when $G_1$ and $G_2$ are independent gamma RVs, a simplified expression of (4) can be obtained as

$$f_{X_1,X_2}(x_1, x_2) = \sum_{h=0}^{\infty} \frac{16(k_1)_h \rho_G^h}{h! \Gamma(k_2+k_1+1)} \frac{k_2-k_1}{(1-\rho_G)^{k_2+1}} \times \frac{1}{(1-\rho_G)^{k_2+1}} \prod_{i=1}^{2} \left[ x_i^{h+1} \exp \left[ -x_i \left( \frac{(1-\rho_G)^h}{\Gamma(k_2 + h)} \right) \right] \right]$$

(6)

The power correlation coefficient between $X_1^2$ and $X_2^2$ is defined as $\rho \triangleq \frac{\text{cov}(X_1^2, X_2^2)}{\sqrt{\text{var}(X_2^2)} \sqrt{\text{var}(X_2^2)}}$ [15, eq. (7.1)], where $\text{cov}(\cdot, \cdot)$ is the covariance and $\text{var}(\cdot)$ is the variance. Using (6) and [3, eq. (2.64)], $\rho$ can be obtained in a simple closed-form expression as

$$\rho = \left( \frac{k_2-1}{k_2+1} \right) \left( 1 + \rho_R \right)$$

(7)

For $\rho_G = 0$ and $k_1 = k_2$, (8) further simplifies to another previous known expression [11, eq. (12)], i.e., the correlated $K$ PDF with identical parameters.

Using (8) and [13, eq. (6.621/3)], the joint CF of $X_1$ and $X_2$ can be expressed as

$$\Phi_{X_1,X_2}(s_1, s_2) = (1-\rho_R)^{k_2} \prod_{i=1}^{2} \left[ \Gamma(k_2) \right]$$

(8)

where $j = \sqrt{-1}$

By expressing $K_i(\cdot)$ in (8) as in [13, eq. (9.34/4)] and using [16, eq. (07.34.210003.01)], the joint CDF of $X_1$ and $X_2$ can be obtained as

$$F_{X_1,X_2}(x_1, x_2) = \sum_{h=0}^{\infty} \rho_R \prod_{i=1}^{2} \left[ \left( \frac{k_2+k_1+1}{1-\rho_R} \right)^{\frac{1}{2}} \right]$$

(9)

where $G_1^{(1,3)}$ is the Meijer’s $G$-function [13, eq. (9.301/)] and $\xi_i = x_i^2/(1-\rho_R)\Omega_i$. Furthermore, for non-integer values of $k_i$, an
alternative and quite simpler expression for the joint CDF can be extracted using [16, eq. (03.04.21.0007.01)] in (8), yielding

\[ F_{X_1, X_2}(x_1, x_2) = \pi^2 (1 - \rho_R) \sum_{\ell=0}^{\infty} \rho_R^\ell \prod_{i=1}^{2} \csc(\pi k_i) \times \left[ \sum_{t=0}^{\ell} \frac{1}{\Gamma(k_i)} F_2(t+1; t+2 - k_i, t + 2; \zeta_i) - \frac{\zeta_i^{k_i}}{t!} F_2(k_i - t, k_i + 1; \zeta_i) \right]. \tag{11} \]

In (11), \( F_{\alpha}(\cdot; \cdot; \cdot) \) represents the regularized generalized hypergeometric function [16, eq. (07.32.02.0001.01)], where \( p \) and \( q \) are integers. This function is related with the well-known generalized hypergeometric function \( F_{\alpha}(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = F_{\alpha}(a_1, \ldots, a_p; b_1, \ldots, b_q; z) / \prod_{k=1}^{q} \Gamma(b_k) \) [16, eq. (07.32.26.0001.01)].

III. SD RECEIVERS OUTAGE PROBABILITY

In principal, the previously derived expressions can be used to obtain the performance of a wide range of diversity receivers operating over fading/shadowing channels modeled by the correlated \( K \)-distribution. However here the OP performance of an SD receiver is studied. Thus, a dual-branch SD receiver operating over an additive white Gaussian noise (AWGN) channel, subject also to flat fading, modeled by the previously discussed \( K \)-distribution, is considered. The instantaneous signal-to-noise ratio (SNR) of the \( i \)th, \((i = 1 \text{ and } 2)\), input branch is given by

\[ \gamma_i = X_i^2 \frac{E_s}{N_0} \tag{12} \]

with \( E_s \) being the transmitted symbol energy, and \( N_0 \) the power spectral density of the AWGN, assumed to be identical for both branches. The corresponding average value of \( \gamma_i \) can be mathematically expressed as

\[ \bar{\gamma}_i = E_X_i \langle \gamma_i^2 \rangle \frac{E_s}{N_0} = k_i^2 \Omega_k E_s / N_0. \tag{13} \]

Moreover, the instantaneous SNR at the output of the SD receiver, \( \gamma_{\text{out}} \), is obtained by selecting the max (\( \gamma_1, \gamma_2 \)) and the CDF of \( \gamma_{\text{out}} \) is given by \( F_{\gamma_{\text{out}}}(\gamma) = F_{\gamma_1, \gamma_2}(\gamma, \gamma) \) [15, eq. (6.54)]. Clearly from (12) and (13), \( X_i^2/\bar{\gamma}_i = k_i^2 \bar{\gamma}_i/\bar{\gamma}_i \), and thus with the aid of this relationship, it is not difficult to recognize that from (11), \( F_{\gamma_1, \gamma_2}(\gamma_1+\gamma_2) \) can be obtained. Hence, by denoting the predefined outage threshold as \( \gamma_{\text{th}} \), the OP at the output of SD can be easily obtained as \( P_{\text{out}}(\gamma_{\text{th}}) = F_{\gamma_{\text{out}}}(\gamma_{\text{th}}) \), leading to

\[ P_{\text{out}}(\gamma_{\text{th}}) = \pi^2 (1 - \rho_R) \sum_{\ell=0}^{\infty} \rho_R^\ell \prod_{i=1}^{2} \csc(\pi k_i) \times \left[ \sum_{t=0}^{\ell} \frac{1}{\Gamma(k_i)} F_2(t+1; t+2 - k_i, t + 2; \zeta_i) - \frac{\zeta_i^{k_i}}{t!} F_2(k_i - t, k_i + 1; \zeta_i) \right]. \tag{14} \]

Furthermore, by using the definition of the generalized hypergeometric function \( F_{\alpha}(a_1, \ldots, a_p; b_1, \ldots, b_q; z) \) [13, eq. (9.14/1)], (15) can be obtained in closed form as

\[ P_{\text{out}}(\gamma_{\text{th}}) = (1 - \rho_R) \sum_{\ell=0}^{\infty} \rho_R^\ell \prod_{i=1}^{2} (\frac{1}{\Gamma(t+2-k_i)(t+1)} - \frac{\zeta_i^{k_i}}{t!} \frac{1}{\Gamma(k_i - t)!} \left[ a_3(1, 1-k_i, 1-k_2; 2, 1-k_1; -\zeta_i \rho_R) - 1 \right] \frac{1}{\Gamma(k_i)} F_2(1-k_2; 2, 2-k_1; -\zeta_i \rho_R) \right] + \frac{\zeta_i^{k_2}}{1-k_1} \frac{1}{\Gamma(k_2)} F_2(1-k_1; 2, 2-k_2; -\zeta_i \rho_R) \right] + \frac{\zeta_i^{k_1}}{1-k_2} \frac{1}{\Gamma(k_1)} F_2(1-k_1; 2, 2-k_2; -\zeta_i \rho_R) \right] + \frac{\zeta_i^{k_1} \zeta_i^{k_2}}{1-k_1} \frac{1}{\Gamma(k_1) \Gamma(k_2)} F_2(1-k_1; 2, 2-k_2; -\zeta_i \rho_R) \right]. \tag{16} \]

IV. NUMERICAL RESULTS

Using the previous presented analysis, the OP performance of a dual-branch SD receiver operating over correlated \( K \) fading channels has been numerically evaluated in terms of the OP (see Fig. 1) and a series convergence rate (see Table I). In Fig. 1, the
OP performance is plotted as a function of the first branch normalized outage threshold, $\gamma_0/\gamma_1$, for several values of $k_1$, $\rho_R$ and for both identical (id) and non-id correlated $K$ fading channels. In the case where unbalanced, non-identical, correlated $K$ fading conditions are considered, it is assumed $\gamma_2 = \gamma_1/\sqrt{e}$ and $k_2 = k_1/2$. It is noted that these values of $\gamma_1$ follow the well-established exponential decaying power delay profile [17]. As expected, the obtained performance results show that as $\gamma_0/\gamma_1$ decreases, the outage performance improves, while the performance also improves as $\rho_R$ decreases and/or $k_1$ increases. As also observed in [8], the best performance is obtained when identical fading channels are considered.

Table I summarizes the number of terms required in (10) to achieve accuracy better than $1\%$ after the truncation of the infinite series. In particular, in this table, the number of required terms versus $\gamma = \gamma_1 = \gamma_2$, for $\rho_R = 0.2, 0.7$, and $k = k_1 = k_2$, is presented. As depicted, the required terms increase as $\gamma$ and/or $\rho_R$ increase and/or $k$ decreases, similarly to other works [8], [18], [19]. It should be noted that similar rates of convergence have been obtained for all the infinite series expressions presented in this paper, i.e., (4)–(9).

**V. CONCLUSIONS**

The bivariate $K$-distribution with arbitrary and not necessarily identical shaping and scaling parameters was presented and studied. Assuming correlation in both the signal envelopes and their average powers, generic infinite series expressions for the most important statistical metrics of this composite distribution were obtained. The derived expressions simplify to previous reported results for special cases. Furthermore, capitalizing on these expressions, the OP performance of dual-branch SD receivers operating over correlated $K$ fading/shadowing channels is analytical studied.

**TABLE I**

<table>
<thead>
<tr>
<th>$\gamma$(dB)</th>
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**REFERENCES**


