DIFFUSION OF ASTEROIDS IN MEAN MOTION RESONANCES

KLEOMENIS TSIGANIS and HARRY VARVOGLIS
Section of Astrophysics, Astronomy and Mechanics, Department of Physics, University of Thessaloniki, 540 06 Thessaloniki, GREECE

ANASTASIOS ANASTASIADIS
Institute for Space Applications and Remote Sensing, National Observatory of Athens, 152 36 P. Penteli, GREECE

Abstract. We study transport in the action space of the planar elliptic restricted three-body problem, for values of the semi-major axis, a, corresponding to the outer asteroid belt ($3.45\text{AU} \leq a \leq 3.90\text{AU}$). Numerical estimates of the local diffusion coefficient, D, are made and the results are presented in the form of 'diffusion maps'. For resonances of order $q \leq 5$ the functional dependence of D on the free eccentricity, e_f, is also studied.

1. Introduction

If an asteroid is located in a mean motion resonance with Jupiter, its orbital elements, especially the eccentricity, e, can be transported to Jupiter-crossing values due to chaotic motion. For resonances closer to Jupiter, such as those placed in the outer asteroid belt (defined here by $3.45\text{AU} \leq a \leq 3.90\text{AU}$), a large fraction of orbits is expected to be chaotic while, at the same time, the eccentricity value needed to cross the orbit of Jupiter is small ($e \sim 0.3$). In the absence of mechanisms which can provide 'shortcuts' to high values of e (such as resonant periodic orbits; see Tsiganis *et al.*, 2001), the 'random walk'-like manner by which the eccentricity grows resembles a diffusion process. Murray & Holman (1997; hereafter MH97) constructed an analytical theory for this 'slow chaos' in outer-belt mean motion resonances, in the framework of the planar elliptic restricted three-body problem (hereafter ERTBP). Their calculations were based on an averaged Hamiltonian with 2 degrees of freedom. In this model, chaos is the result of the overlap among the terms of the *resonance multiplet*, which appear explicitly in the expansion of the disturbing function of the ERTBP. Their results indicate extended chaos in resonances of order $q \leq 5$. Their estimate for the *coefficient of diffusion* inside a given resonance, $D(I)$, in the action $I = \sqrt{(1-\mu)\,a}\,e_f^2/2$ (e_f=free eccentricity), reads $D(I) \sim I^p$, with $p = q$ for $e_f \approx e' = 0.048$ (primed elements refer to Jupiter). $D(I)$ can be used to solve the associated transport equation and derive realistic estimates for the mean time that an asteroid takes to escape from the belt.

In this paper we present results on numerical estimates for $D(I)$, based on short-time numerical integrations in the framework of ERTBP. Jupiter was set

Celestial Mechanics and Dynamical Astronomy **00:** 1–5, 2001.

at $a' = 5.2\,\mathrm{AU}$, $e' = 0.048$, $\varpi' = 0$ and $M'_0 = 0$ (ϖ=longitude of perihelion, M=mean anomaly). Two kinds of calculation were made, destined for two different applications.

2. Numerical estimates of D

The first application was to obtain a global view of transport in the outer belt, by measuring D as a function of both a and e_f. Similar *dynamical maps*, constructed for several regions of the solar system and concerning other indices of transport, such as the variation of the proper frequencies, have already been published (see Ferraz-Mello, 1999; Robutel & Laskar, 2001 and Michtchenko & Ferraz-Mello, 2001). For this purpose the outer belt was covered by a grid of 150×30 initial conditions, with $3.45 \leq a \leq 3.90$ and $0 \leq e_f \leq 0.3$ ($\Delta a = 0.003\mathrm{AU}$, $\Delta e_f = 0.01$). All orbits had $\varpi_0 = \varpi' = 0$ and $M_0 = 0$. The integration time was set to $t_{\mathrm{int}} = 10^5$ years. As the period of revolution of ϖ is $P_{\mathrm{sec}} = \dot{\varpi}^{-1} < 10^4$ years in the outer belt, t_{int} corresponds to $t > 10P_{\mathrm{sec}}$. The initial osculating eccentricity of each orbit was given by $e_0 = e_f + e_{\mathrm{for}}$, where $e_{\mathrm{for}} = e'\, b^{(2)}_{3/2}(\alpha)/b^{(1)}_{3/2}(\alpha)$ is the forced eccentricity at $\alpha = a/a'$ and $b^j_s(\alpha)$'s are Laplace coefficients. Using an adjacent-average method, short-periodic variations were filtered out from the elements of each orbit. A time-series for e_f was then computed, by taking $e_f(t_i) = (k_{\max} - k_{\min})^{i+N/2}_{i-N/2}/2$ within a running window of length N corresponding to 10^4 years ($k = e\sin\varpi$). Calculating the mean squared displacement in I, $\langle(\Delta I)^2\rangle$, as a function of time, the diffusion coefficient is defined by

$$D(I_0) = \lim_{t\to\infty} \frac{\langle(\Delta I)^2\rangle}{t} \qquad (1)$$

A single orbit would be enough to estimate $D(a_0, e_{f,0})$, $\langle\cdot\rangle$ being a time-average, provided that the ergodic hypothesis holds in the conjugate angles' space. Keeping in mind that this is *not* the case, at least for small eccentricities, we computed *single-orbit estimates* of $D(a, e_f)$, where $\langle(\Delta I)^2\rangle = \langle[I(t_i+\tau) - I(t_i)]^2\rangle_i$. The value of D was then taken, at each cell, as the slope (given by a linear least-squares fit) of the $\langle(\Delta I)^2\rangle(\tau)$ vs. τ curve.

The second application we had in mind was to check the analytical results of MH97, concerning the functional form of $D(I)$ for different resonances. Setting $a = a_{\mathrm{res}} \approx (p/(p+q))^{2/3}$, we split the free eccentricity range $0 \leq e_f \leq 0.25$ into 25 cells. 196 orbits were placed in each cell, with the free pericenter longitude, ϖ_f, and the resonant angle $\psi = p\lambda - (p+q)\lambda'$ (λ=mean longitude) set on a 14×14 grid in $(0, 2\pi) \times (0, 2\pi)$. The diffusion coefficient $D(I)$ was again taken, for each initial value of $I = I_0$, as the slope of $\langle(\Delta I)^2\rangle(t_i)$, only this time $\langle\cdot\rangle$ denotes the *ensemble average* over the conjugate angles, i.e. $\langle(\Delta I)^2\rangle(t) = \langle[I(t) - I(0)]^2\rangle_{(\psi,\varpi_f)}$. We note here that all orbits of the ensemble should, in principle, be chaotic. We do

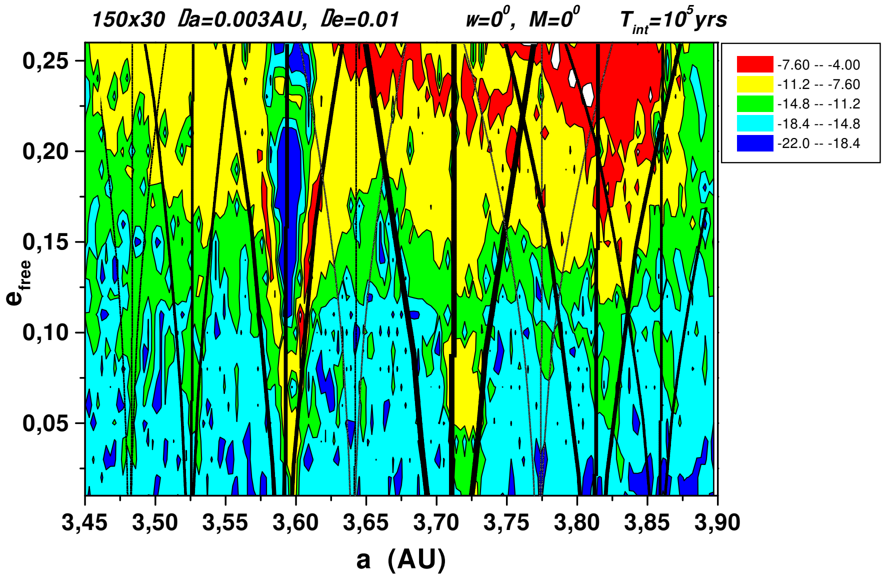

Figure 1. Diffusion in e_f for the outer asteroid belt, in the framework of the ERTBP. The value of $\log D(I)$ (units are in $5.2AU/yr$) is color-coded as shown on the top-right frame. For $e_f > 0.15$ resonances begin to overlap, as their separatrices (bold lines) begin to join each other. The width of the resonant lines is inversely proportional to the order of the resonance.

not know *a priori* whether regular trajectories, or how many, are included within each cell. Thus, this result is in fact an *effective* diffusion coefficient, weighted by the probability of an orbit being chaotic at $I = I_0$.

3. Results

Figure 1 shows a representative diffusion map for the outer belt. The different colors indicate the speed of diffusion, i.e. the value of $\log D_I(a, e_f)$. The separatrices of the lowest-order ($q \leq 5$) resonances, as these are calculated in the integrable pendulum approximation, are superimposed. For $e_f < 0.1$ only “thin” high-diffusivity regions are found, corresponding to the location of these resonances. It is easy to see that adjacent resonances begin to overlap for $e_f > 0.15$ and the value of $D(a, e_f)$ becomes high throughout the whole range of a. Note the low-diffusivity area centered at $a \approx 3.59$AU, where the 7/4 resonance, the most chaotic resonance of the outer belt, is located. This result is related to our selection of initial phases ($\varpi = 0$, $M = 0$), which corresponds to the stable stationary point of $\sigma = 7\lambda' - 4\lambda - 3\varpi = 0$ (see also Robutel & Laskar 2001). For resonances with $q \geq 5$, the $e_f < 0.1$ region seems to be covered by regular orbits, as the value of D is not very different from that of the background, ‘non-resonant’, domain.

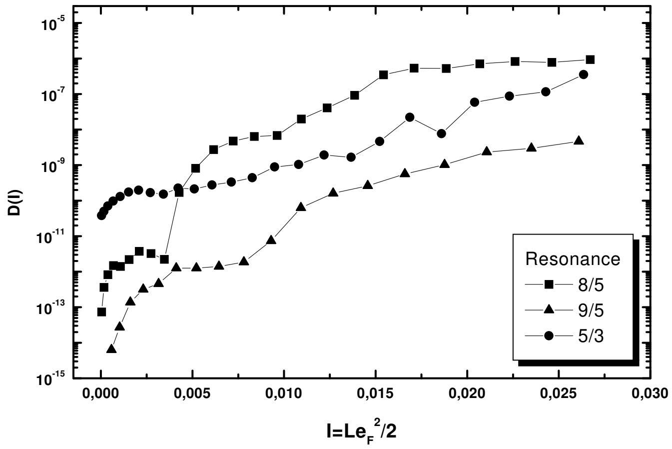

Figure 2. $D(I)$ as a function of I for the 5/3, 8/5 and 9/5 mean motion resonances. The $y-$scale is logarithmic. The form of the $D(I)$ curve is clearly depending on the resonance under question.

Figure 2 shows the $D(I)$ curves for the 5/3, 8/5 and 9/5 resonances. Note the "stair-like" behavior of these curves, for small eccentricities. This is predicted by the theory of MH97, as for different values of e_f the strongest term of the resonant multiplet, which determines the exponent p in $D(I) \sim I^p$, is different. The small-eccentricity part of these curves can indeed be fitted by functions of the form $D(I) \propto I^b$, but for $e_f > 0.2$ this picture does not hold. We emphasize though that our results are not yet complete and more detailed computations are needed in order to perform reliable fits. If we go to $e_f > 0.25$, the curves tend to converge to a single value of $D = D_{QL}$, the *quasi-linear limit*, which is approached as all resonances begin to overlap.

4. Conclusions

Dynamical maps, such as the one presented above, are useful in getting a global view of the dynamics within wide areas of the orbital elements' space. Our results indicate that chaotic diffusion is prominent in the outer asteroid belt, especially for $e_f > 0.15$. In this respect, only asteroids with free eccentricities smaller than this value can be expected to reside in this region. We note though that, in order to obtain a more complete picture of the global dynamics, the map has to account for different initial values of the conjugate angles. Thus, either D has to be computed as an ensemble average in each cell (clearly a cumbersome computation) or, at least, one has to compute maps, like the one shown in Fig. 1, for several initial values of ϖ and M.

For $e_f < 0.15$ a single-resonance approximation is valid in the outer belt. Our numerical results seem to agree with the analytic results of MH97, although more detailed computations are needed. Further results on outer-belt resonances, as well as a detailed analysis of the computational methods, is under way and will be presented in a forthcoming paper.

Acknowledgements

K. Tsiganis wishes to acknowledge financial support by the LOC of *CELMEC III*, which enabled his participation in this meeting. The work of *K.T.* was supported by the State Scholarships Foundation of Greece (**IKY**).

References

Ferraz-Mello, S.: 1999, Slow and Fast Diffusion in Asteroid-Belt Resonances: A Review, *Celest. Mech. Dynam. Astron.* **73**, 25–37.

Michtchenko, T.A., and Ferraz-Mello, S.: 2001, Resonant Structure of the Outer Solar System in the Neighborhood of the Planets, *Astron. J* **122**, 474–481.

Murray, N., and Holman, M.: 1997, Diffusive chaos in the outer asteroid belt, *Astron. J.* **114**, 1246–1259.

Robutel, P., and Laskar, J.: 2001, Frequency Map and Global Dynamics in the Solar System I, *Icarus* **152**, 4–28.

Tsiganis, K., Varvoglis, H., and Hadjidemetriou, J.D.: 2001, Stable chaos in high-order Jovian resonances, *Icarus*, in press.