Transport in the outer asteroid belt: Fokker-Planck approach vs. numerical integration*

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ABSTRACT

In this paper we examine the statistical properties of transport, for orbits of fictitious asteroids initiated in three outer-belt resonances: the 7:4, 9:5 and 12:7 mean motion resonances of the 2-D elliptic restricted three-body problem. Two alternative approaches are used: (i) numerical integration of distributions of initial conditions and (ii) simulation of the diffusion of the eccentricity-related action, through the numerical solution of a 1-D Fokker-Planck equation. The diffusion coefficient, *D*, is determined numerically, using the definition of "local transport coefficients". The statistical results of (i) and (ii) are compared. Qualitative agreement - and even quantitative in some cases - between the two approaches is found. For the improvement of the efficiency of (ii) proper modifications, concerning mostly the calculation of *D(I)*, are needed.

INTRODUCTION – MOTIVATION

The possibility of formulating a statistical description of the motion of asteroids, at least in regions where chaotic motion dominates, has received little attention so far. It is known that chaotic motion results in a slow diffusion-like evolution of the action variables. Thus, the evolution of any initial distribution function of the actions could, in principle, be described by the solution of a *Fokker-Planck-Kolmogorov* (FPK) equation, provided that an appropriate calculation of the *diffusion coefficient*, *D*, could be achieved. Varvoglis & Anastasiadis (1996) managed to reproduce the statistical 'law of escape' of Lecar et al. (1992) by solving the FPK equation (Eq. 1) with $D=a\lambda^b$ (λ=Lyapunov exponent; Konishi 1989). This result holds only in the resonance-overlap regime of the 2-D elliptic restricted three-body problem.

$$\frac{\partial f(I,t)}{\partial t} = D\frac{\partial^2 f(I,t)}{\partial I^2} - \frac{f(I,t)}{T_E} \qquad (1)$$

Murray & Holman (1997, hereafter M&H), working on the planar elliptic three-body problem, derived an analytical approximation of the diffusion coefficient (in the quasi-linear approximation) for a single-resonance (of order *q*) domain. They used an approximate Hamiltonian derived by expanding the disturbing function and keeping only the resonant term and the lowest-order secular term. The coefficient then becomes action-dependent:

$$D_{QL}(I) = \frac{\langle(\Delta I)^2\rangle}{T_L} = \frac{1}{2} p_0^2 \tilde{\phi}_0^2 (I,e')\frac{\pi}{\mu A} \propto I^q \qquad (2)$$

Here *e'* is the eccentricity of Jupiter's orbit, and $I=Le^2/2$ is the eccentricity-related action. Also, $T_L=1/\lambda$ is the Lyapunov time, μ is the mass ratio of Jupiter to the total mass of the system, *A* is a function of the semi-major axes *a* and *a'*, p_0 ($0\leq p_0\leq q$) is the integer coefficient denoting the strongest sub-resonance[1] and ϕ_0 is the coefficient of the corresponding term in the expansion of the disturbing function. This formalism, although elegant, is still a gross approximation. Eq. (2) is valid only for $e<e_c/q$ ($e<0.1$ for $q\approx 4$-6, e_c being the critical planet-crossing eccentricity for some value of the semi-major axis, a). Also, it does not account for resonance-overlap (which may occur at small eccentricities in the outer belt; see Tsiganis et al. 1999). Moreover, the extension of this formalism so as to include the perturbations induced by other planets is not an easy task. Finally, the validity of the quasi-linear approximation can be questioned since, in phase-space domains where regions of regular motion still persist, the time needed for the 'local' diffusion coefficient, $D(I)$, to saturate may be much larger than T_L (Yannacopoulos & Rowlands, 1997).

In this paper we derive the 'local' diffusion coefficients for three outer-belt resonances (7:4, 9:5 and 12:7) in the framework of the planar elliptic three-body problem. The procedure used is described in the next section. The results obtained from the solution of the corresponding FPK equation are then presented and compared to the results obtained from long-time numerical integration. Finally, we discuss the problems encountered in this study and suggest possible ways of improving the efficiency of the statistical formulation.

NUMERICAL CALCULATION OF *D(I)*

The calculation of $D(I)$ is made as follows. For each resonance the eccentricity range $e\in(0,0.4)$ is split into 50 bins. At each bin 500 orbits are set, with ϖ and M (ϖ = longitude of perihelion, M = mean anomaly) chosen at random in $(0,2\pi)$. All orbits have initially $a=a_{\rm res}$. The integration is done using the *swift_rmvs3* symplectic integrator from the **SWIFT** package of Levison & Duncan (1994). The total integration time is 1,000-10,000 years and the time-step is $\delta t=36.525$ days. Figure (1a) is a plot of $D(I)$ for the three resonances studied. In the low-I (low-eccentricity) regime the results yield $D_{7:4}>D_{12:7}>D_{9:5}$. This does not agree with M&H in what concerns the 9:5 and 12:7 resonances. However it reflects the results of Tsiganis et al (1999), which have shown that escape from the vicinity of the 12:7 resonance is controlled by the overlap of this resonance with adjacent low-order resonances (7:4 and 5:3) and is, therefore, much faster than expected (from M&H). In the high-I regime, all three coefficients have similar values, and this is due to the extensive resonance overlap, which governs the dynamics of the high-eccentricity regions of the outer belt.

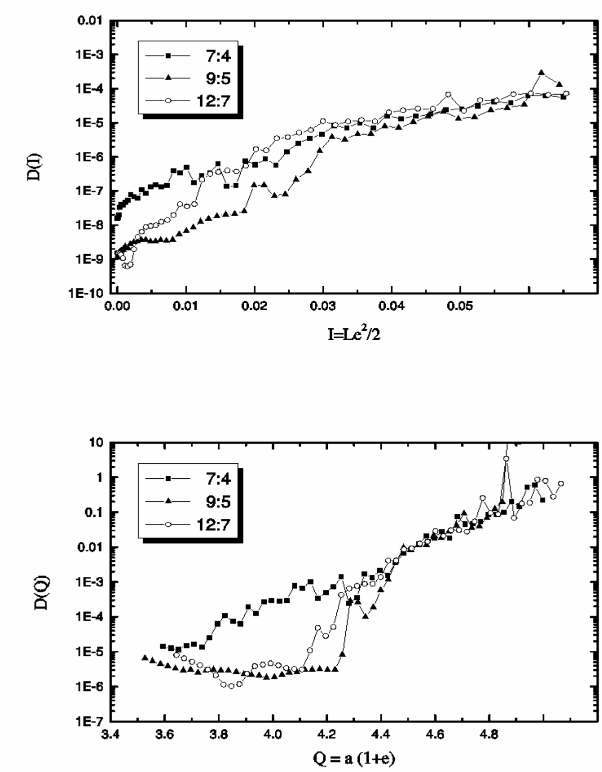

Fig. 1: Diffusion coefficients *D(I)* (top) and *D(Q)* (bottom). Note the 'jump' in *D(Q)* at Q≈4.2 AU.

[1] In the elliptic problem every mean motion resonance of order q splits into $q+1$ 'sub-resonances'

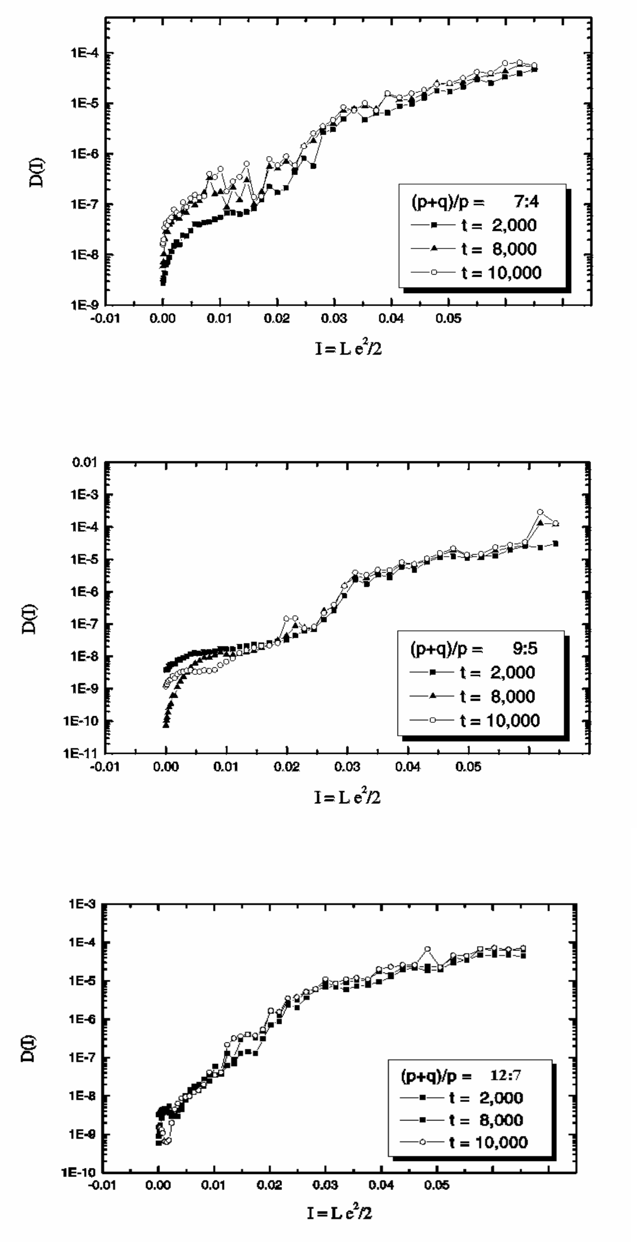

Fig. 2: Variation of D(I) with time. Differences up to an order of magnitude occur at some points (the y-scale is logarithmic).

This can also be seen in Fig. (1b), where we have plotted $D(Q)=\langle(\Delta Q)^2\rangle/t$ instead of $D(I)$, $Q=a(1+e)$ denoting the apocentric distance, again for all three cases. A 'jump' is observed in $D_{12:7}$ and $D_{9:5}$ at $Q\approx4.2$ AU. This distance defines roughly the lower bound of the 'resonance-overlap' regime, as predicted (for nearly circular orbits) by Wisdom's '$\mu^{2/7}$-law' (Wisdom, 1980), in the restricted problem.

Unfortunately, the integration time (10^4 years) is too small for $D(I)$ to saturate in the whole range of I. We note that $T_L\approx10^4$ years for the outer belt, i.e. $D(I)=D_{QL}(I)$. For the 7:4 resonance the results are much better than for the other two cases, where large variations are seen (Figure 2) in the low-action regime – which is the most significant range for a correct estimation of the escape time-scale. In order to use $D(I)$ for the solution of the FPK equation a continuous function needs to be fitted to the data; otherwise the grid has to be generated according to the number of bins used for calculating $D(I)$. The 7:4 data are fitted very well by a power-law, $D(I)=aI^b$, with $b=3.3\pm0.3$.

On the other hand, the 9:5 and 12:7 data are not well fitted by simple power-laws, as the resulting coefficients (a,b) have very large error-bars. However, we decided to use these fits in order to get a first approximation of the corresponding FPK-solution.

SOLUTION OF THE FPK-EQUATION

We use the calculated coefficients for the numerical solution of the FPK equation. We note that a 1-D approach is justified, since numerical experiments have shown that the increase of the apocentric distance of an asteroid's orbit (which leads to close encounters) is dominated by the eccentricity increase, and not by the increase of its semi-major axis. For a non-constant D, the 1-D form of the FPK equation is

$$\frac{\partial f(I,t)}{\partial t}=\frac{\partial}{\partial I}\left(\frac{D(I)}{2}\frac{\partial f(I,t)}{\partial I}\right) \qquad (3)$$

and the boundary conditions used

$$\left.\frac{\partial f(I,t)}{\partial I}\right|_{I=0} = 0, \quad \forall t$$
$$\left. f(I,t)\right|_{I=I_c} = 0, \quad \forall t \qquad (4)$$

correspond to a *reflecting* boundary at I=0 (e=0) and an *absorbing* boundary at $I=I_c$ (I_c corresponds to e_c, the critical eccentricity that leads to a Jupiter-crossing orbit). We use a standard *explicit scheme* to derive the discrete form of Eq. (3). This scheme is $\boldsymbol{O}(\Delta I^2, \Delta t)$ accurate and is subject to a *stability criterion*, $\Delta t<2(\Delta I)^2/D_i$, that has to be met on every grid point, i. Finally, the initial distribution was chosen to represent a set of orbits with randomly chosen initial eccentricities in the range (0.05, 0.1).

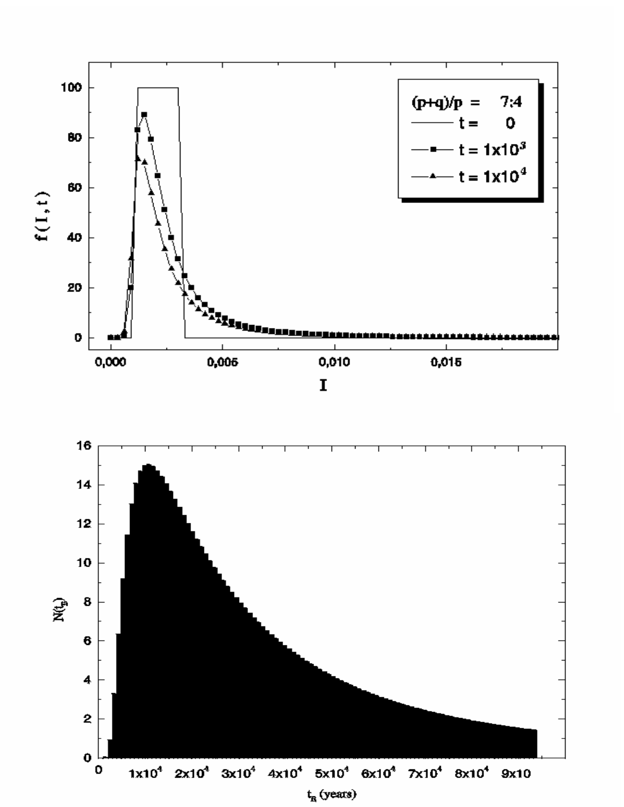

Fig. 3: The FPK solution for the 7:4 resonance (top) and the corresponding histogram of escape times (bottom).

The solution for the case of the 7:4 resonance is shown in Fig. (3). The top frame shows the time evolution of the initial distribution *f(I,t)*. It is evident that the resulting distribution is highly non-symmetric, and this is due to the functional form of *D(I)*. Note that orbits with *e>0.1* do not 'survive'. The bottom frame is the histogram of escape times. If D was constant, the histogram should have the shape of a Gaussian distribution. Instead, it is an asymmetric log-normal distribution, i.e. it possesses a long 'tail' of 'late escapers'. Qualitatively, the solution is the same for the other two cases also. Since a 'mean' value of the escape time is not very meaningful for such a distribution, we use the median of the distribution to characterise the escape time-scale. For the 7:4 case we have $T^m_{7:4} \approx 25{,}000$ yrs. For the other two cases the results yield $T^m_{9:5} \approx 2.5$ Myrs and $T^m_{12:7} \approx 100{,}000$ yrs.

COMPARISON WITH NUMERICAL INTERGRATIONS

We also integrated a distribution of 200 'asteroids' set on each resonance (600 particles in total), with initial conditions $a=a_{res}$, $e \in (0.05, 0.1)$ and ϖ, M chosen again at random. A particle was considered to be ejected, if it approached Jupiter within Hill's sphere (close encounter).

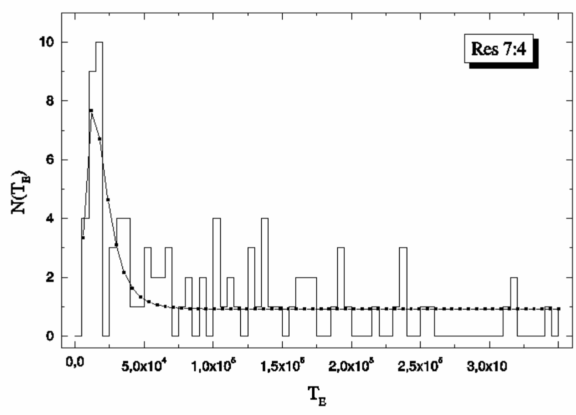

Fig. 4: The escaping profile for the 7:4 resonance, as calculated from a 1Myr numerical integration.

The resulting escape time histograms are qualitatively similar to those derived from the FPK-solution, but a detailed comparison cannot be made for such a small number of orbits. The values obtained for the median escape times are: $T^{m}_{7:4} \approx 30{,}000$ yrs, $T^{m}_{9:5} \approx 15$ Myrs and $T^{m}_{12:7} \approx 6.5$ Myrs. The results for the 7:4 resonance (Fig. 4) are in perfect agreement with the FPK-solution. For the 9:5 case, the median escape times do not match, but it is encouraging that the values are of the same order of magnitude. The results for the 12:7 case are evidently not in agreement. Although such a discrepancy was not expected, it can be explained in terms of the complicated phase-space structure in the vicinity of the 12:7 resonance, as shown in Tsiganis et al (1999).

CONCLUSIONS

In this paper we derived the statistics of escape for three outer-belt mean motion resonances of different order by applying the diffusive approximation. A numerical method for estimating the 'local' diffusion coefficient, *D(I)*, was presented and the numerical solution of the corresponding FPK equation was given. Finally, these results were compared to the statistics as taken from long-time numerical integrations. The results show that the quasi-linear approximation of *D(I)* can give not only qualitatively but also quantitatively good results, for regions where chaotic motion dominates (the 7:4 case). However, it is clear that for 'less chaotic' domains (the 9:5 and 12:7 cases) this approximation does not hold. This is, to our opinion, not a failure of the diffusive approximation but, rather, a matter of insufficient computational effort. In fact, large fluctuations of *D(I)* with time, especially in the low-*I* regime, imply that longer integration times are needed in order for *D(I)* to be representative of the asymptotic behaviour of the actions. However, other improvements are also needed in order to make sure that *D(I)* is properly calculated.

The long-term evolution of *f(I)* is directly related to the *secular* evolution of the actions. Thus, high-frequency (non-resonant) fluctuations, which are present in the calculation of an asteroid's orbital elements, are not related to the secular growth of the orbital elements, being however superimposed to the value of *D(I)* as 'noise' (especially those involving the semi-major axis, *a*). For short integration times and for a few-particle covering of the eccentricity bins (500 orbits/bin may prove to be inadequate), the 'amplitude' of this noise may not be negligible. Increasing the number of orbits/bin is a very expensive solution. The best, probably, way to circumvent this problem is to perform an *averaging* of the orbital elements over short-period terms, before calculating *D(I)*. In this way, only the resonant and secular terms are taken into account, without however using a 'truncated' Hamiltonian from the start, as in M&H.

Finally, we would like to comment on the numerical scheme used to solve the FPK equation. The accuracy of the scheme is not a matter of great importance, since improving the accuracy significantly with minimum computational cost is not easy for such problems. However, the stability criterion forced by the explicit formulation may render the scheme inefficient when calculating the evolution of *f(I)* for $t \approx 5$ Gyrs. This is because of the functional form of *D(I)*, which is a steeply growing function of *I*. Thus, the criterion imposes a very small time-step, since otherwise it would not be respected in the high-*I* part of the action interval. However, the important part for us is that of the low-*I* values. Therefore, other schemes (implicit) should be used in order to avoid unnecessarily long computations.

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