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Kinetic Description of Particle Interaction with a Gravitational Wave

A. Anastasiadis,[1] K. Kleidis[1] and H. Varvoglis[1]

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The interaction of charged particles, moving in a uniform magnetic field, with a plane polarized gravitational wave is considered using the Fokker–Planck–Kolmogorov (FPK) approach. By using a stochasticity criterion, we determine the exact locations in phase space, where resonance overlapping occurs. We investigate the diffusion of orbits around each primary resonance of order m by deriving general analytical expressions for an effective diffusion coefficient. A solution of the corresponding diffusion equation (Fokker–Planck equation) for the static case is found. Numerical integration of the full equations of motion and subsequent calculation of the diffusion coefficient verifies the analytical results.

KEY WORDS : Fokker–Planck equation with magnetic field

1. INTRODUCTION

The many efforts that have been made to detect gravitational waves have so far given no convincing evidence that they have actually been seen [1]. This is due to the fact that not only is their amplitude very small [2], but it is highly possible that some kind of damping mechanism operates on them as they travel through space [3–5]. This damping may originate in the interaction of the gravitational wave with interstellar matter [6,7].

In a recent paper [8], hereafter is referred to as Paper I, the problem of the interaction of a charged particle with a gravitational wave, in the

[1] Section of Astrophysics, Astronomy and Mechanics, Department of Physics, Aristotle University of Thessaloniki, GR-54006 Thessaloniki, Greece

presence of a uniform magnetic field, has been considered for various directions of propagation of the wave with respect to the magnetic field. It was found that in the oblique propagation case the motion of the particle becomes chaotic and may be considered as a diffusion in momentum space, provided that its initial momentum is sufficiently large.

In order to address in detail the interaction of charged particles with a gravitational wave, one should try to calculate the diffusion rate (in momentum space) of the particles which follow chaotic trajectories. This task involves the derivation of a Fokker–Planck (FP) type diffusion equation and the calculation of the corresponding diffusion coefficient [9].

In the present paper we investigate the energy diffusion of charged particles in the presence of a uniform magnetic field, $\vec{B} = B_0 \hat{e}_z$, due to their non-linear interaction with a linearly polarized gravitational wave, propagating obliquely with respect to the direction of the magnetic field ($20° \leq \theta \leq 60°$). The analysis is carried out in the framework of the weak field theory, considering the gravitational wave as a small pertubation in a flat space time. We use the Fokker–Planck–Kolmogorov (FPK) approach and refer to the globally stochastic regime, where overlapping of many resonances occurs. In a partially stochastic regime the FPK approach cannot be applied, as the particles do not undergo "normal diffusion" (random walk process) but rather follow Levy statistics [10]. This statistical approach is possible only after deriving general formulas that hold for every value of the perpendicular energy of the charged particle and not just for the highest values (the simplified case that has been considered in Paper I).

The motion of a charged particle in curved spacetime is given, in Hamiltonian formalism [11], by the differential equations

$$\frac{dx^\mu}{d\lambda} = \frac{\partial H}{\partial \pi_\mu}, \qquad \frac{d\pi_\mu}{d\lambda} = -\frac{\partial H}{\partial x^\mu}, \tag{1}$$

where π_μ are the generalized momenta (corresponding to the coordinates x^μ) and the super-Hamiltonian H is given by the relation

$$H = \tfrac{1}{2} g^{\mu\upsilon}(\pi_\mu - eA_\mu)(\pi_\upsilon - eA_\upsilon) \equiv \tfrac{1}{2} \tag{2}$$

(in a system of geometrical units where $\hbar = c = G = 1$). In eq. (2) $g^{\mu\upsilon}$ denotes the components of the contravariant metric tensor, which are defined as

$$g^{\mu\upsilon} = \eta^{\mu\upsilon} + h^{\mu\upsilon}, \tag{3}$$

with $\eta^{\mu\upsilon} = \mathrm{diag}(1, -1, -1, -1)$ and $|h^{\mu\upsilon}| \ll 1$. A_μ is the vector potential, corresponding to the tensor $F_{\mu\upsilon}$ of the electromagnetic field in a curved spacetime. The mass of the particle is taken equal to 1. For the specific form of the magnetic field we may take

$$A^0 = A^1 = A^3 = 0, \qquad A^2 = B_0 x^1 . \tag{4}$$

2. THE STOCHASTICITY CRITERION

We consider the case of a charged particle moving in the curved spacetime background of a linearly polarized gravitational wave, which propagates obliquely with respect to the direction of a uniform and static, in time, magnetic field, $\vec{B} = B_0 \hat{e}_z$. The non-zero components of the metric tensor are presented in Paper I (see references therein) and we normalize lengths and time to c/Ω, where Ω is the Larmor angular frequency. Furthermore we eliminate one degree of freedom from our dynamical system through the canonical transformation

$$\begin{aligned} \vartheta^3 &= x^0 - \cos\theta\, x^3, \qquad \pi_3 = -\cos\theta\, I_3 , \\ \vartheta^0 &= x^0, \qquad \pi_0 = I_3 + I_0 . \end{aligned} \tag{5}$$

Accordingly, the problem of the motion of a charged particle in a gravitational wave is reduced to a two-degrees of freedom dynamical system [8], and the super-Hamiltonian (2), in this case, is written in the form

$$\begin{aligned} H = {} & \frac{1}{2} I_3^2 - \frac{1}{2} \frac{1 + \alpha \sin^2\theta \sin(\upsilon\Theta)}{1 + \alpha \sin(\upsilon\Theta)} \pi_1^2 \\ & - \frac{1}{2} \frac{(x^1)^2}{1 - \alpha \sin(\upsilon\Theta)} - \frac{1}{2} \frac{1 + \alpha \cos^2\theta \sin(\upsilon\Theta)}{1 + \alpha \sin(\upsilon\Theta)} \cos^2\theta I_3^2 \\ & + \frac{1}{2} \frac{\alpha \sin 2\theta \cos\theta \sin(\upsilon\Theta)}{1 + \alpha \sin(\upsilon\Theta)} \pi_1 I_3 , \end{aligned} \tag{6}$$

where we have set

$$\Theta = \sin\theta\, x^1 - \vartheta^3 . \tag{7}$$

In eq. (6) α is the normalized, dimensionless amplitude of the gravitational wave and $\upsilon = \omega/\Omega$ denotes the dimensionless frequency.

The dynamical system under consideration possesses chaotic regions in phase space when $\alpha \neq 0$ [7,8,12]. In order to examine the transition from regular to stochastic motion we use *Chirikov's overlap criterion* [13,14] to obtain the lowest amplitude of the gravitational wave, α_{thr}, above which the dynamical system shows prominent chaotic behaviour.

We first write the Hamiltonian (6) in *action-angle* variables through the canonical transformation

$$x^1 = (2I_1)^{1/2} \sin \vartheta^1, \qquad \pi_1 = (2I_1)^{1/2} \cos \vartheta^1, \tag{8}$$

and, since $\alpha \ll 1$, we make the approximation

$$\frac{1}{1 \pm \alpha \sin(\upsilon\Theta)} \approx 1 \mp \alpha \sin(\upsilon\Theta). \tag{9}$$

The resulting Hamiltonian is of the form

$$H = H_0 + \alpha H_1 \sin(\upsilon\Theta). \tag{10}$$

We expand the trigonometric term of the perturbation in a Fourier series [7]. After further manipulation, the Hamiltonian (10) is written in the form

$$\begin{aligned} H = {} & \frac{1}{2} \sin^2 \theta I_3^2 - I_1 \\ & + \frac{\alpha}{2} \Bigg[- I_1 \sin^2 \theta \sum_{\ell=-\infty}^{\infty} J_\ell(\upsilon r) \sin(\ell\vartheta^1 - \upsilon\vartheta^3) \\ & + I_1 (1 + \cos^2 \theta) \sum_{\ell=-\infty}^{\infty} [2J''_\ell(\upsilon r) + J_\ell(\upsilon r)] \sin(\ell\vartheta^1 - \upsilon\vartheta^3) \\ & + \cos^2 \theta \sin^2 \theta I_3^2 \sum_{\ell=-\infty}^{\infty} J_\ell(\upsilon r) \sin(\ell\vartheta^1 - \upsilon\vartheta^3) \\ & + I_3 \sin 2\theta \cos \theta \sum_{\ell=-\infty}^{\infty} \frac{\ell}{\upsilon} J_\ell(\upsilon r) \sin(\ell\vartheta^1 - \upsilon\vartheta^3) \Bigg] \end{aligned} \tag{11}$$

where $J_\ell(\xi)$ is the Bessel function of order ℓ, $r = (2I_1)^{1/2} \sin \theta$ is the linear momentum along the x-axis and a prime denotes differentiation with respect to $\xi = \upsilon r$. The perturbation term of the Hamiltonian function H_1 depends on an infinite series of linear combinations of the angles ϑ^1 and ϑ^3, a fact that leads to resonances. In this case, Chirikov's criterion states that chaos appears when the width of a resonance, δI_1, becomes larger than or equal to the distance between two consecutive first order resonances, ΔI_1.

By a near identity transformation we remove all trigonometric terms from H_1, except from the one of order $\ell = m$, which generates the principal resonance and corresponds to the family of islands whose width enters in the stochasticity criterion [7]. The resulting Hamiltonian contains only the integrable part H_0 and the dominant term and it is therefore called the *resonant Hamiltonian*, H_R [15,16]. Performing the canonical transformation

$$\vartheta^{1*} = \vartheta^1 - \frac{\upsilon}{m}\vartheta^3, \qquad I_1^* = I_1, \qquad \vartheta^{3*} = \vartheta^3, \qquad I_3^* = I_3 + \frac{\upsilon}{m} I_1, \tag{12}$$

the resonant Hamiltonian is finally written in the form

$$\begin{aligned} H_R = {} & \frac{1}{2}\sin^2\theta\left(I_3^* - \frac{\upsilon}{m}I_1\right)^2 - I_1 \\ & - \frac{\alpha}{2}\Bigg[I_1\sin^2\theta J_m(\xi) - I_1(1+\cos^2\theta)\left[2J_m''(\xi) + J_m(\xi)\right] \\ & - \cos^2\theta\sin^2\theta\left(I_3^* - \frac{\upsilon}{m}I_1\right)^2 J_m(\xi) \\ & + 2\left(I_3^* - \frac{\upsilon}{m}I_1\right)\cos^2\theta\frac{m}{\upsilon}J_m(\xi)\Bigg]\sin(m\vartheta^{1*}). \end{aligned} \tag{13}$$

Since ϑ^3 is a cyclic coordinate, the corresponding generalized momentum I_3^* will be a constant of the motion, so that the dynamical system has one degree of freedom. Hamiltonian (13) describes the motion of a particle around each first order resonance. Using the resonant condition

$$m\frac{d\vartheta^1}{d\lambda} = \upsilon\frac{d\vartheta^3}{d\lambda} \tag{14}$$

and the fact that $H_0 \approx \frac{1}{2}$, we find the order m of the dominant resonance,

$$m = \upsilon(1 + 2I_1)^{1/2}\sin\theta. \tag{15}$$

In this case, $m \neq \upsilon r$ and not $m \simeq \upsilon r$, which was the case considered in Paper I, for $I_1 \gg 1$. This is because in the present paper we are interested in a general formula for the stochasticity threshold, valid for every I_1. The distance ΔI_1 between two consecutive first order resonances is calculated by eq. (13) and the fact that $\Delta m = 1$,

$$\Delta I_1 = \frac{(1+2I_1)^{1/2}}{\upsilon\sin\theta}, \tag{16}$$

while the corresponding *resonant width* δI_1 is given by [15,16]

$$\delta I_1 = \Bigg[\frac{8\alpha m^2}{\upsilon^4\sin^4\theta}\,|\,(m^2 - \upsilon^2\sin^2\theta)(1+\cos^2\theta)J_m'' + (4m^2 - \upsilon^2\sin^2\theta)\cos^2\theta J_m\,|\Bigg]^{1/2} \tag{17}$$

Then Chirikov's criterion, $\delta I_1 \geq \Delta I_1$, reads

$$\frac{1}{\alpha} \leq 8|(m^2 - \upsilon^2 \sin^2 \theta)(1 + \cos^2 \theta)J_m'' + (4m^2 - \upsilon^2 \sin^2 \theta)\cos^2 \theta J_m| . \quad (18)$$

The above relation is the most general form of the stochasticity criterion and holds for any value of I_1, υ and θ. We see that for $I_1 \gg 1$ it reduces to

$$8\upsilon^2 r^2 |4\cos^2 \theta J_m + (1 + \cos^2 \theta)J_m''| \geq \frac{1}{\alpha}, \quad (19)$$

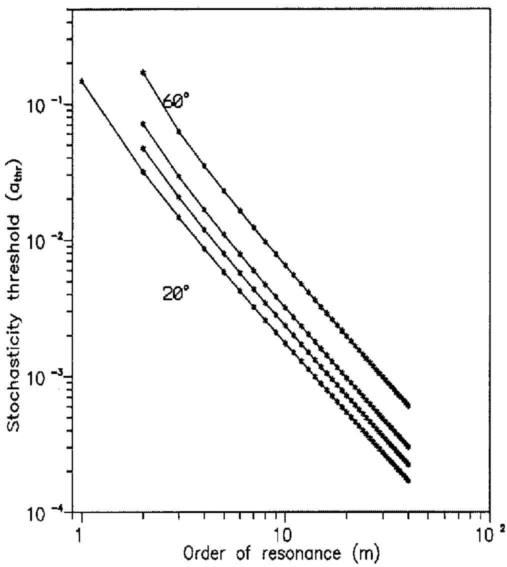

Figure 1. The stochasticity threshold α_{thr} versus the order of resonance m for different values of the wave propagation angle θ and $\upsilon = 1.8$

which is the corresponding result of Paper I. In this approximation, we may obtain an asymptotic form of the stochasticity threshold, by taking $r \to \infty$ [8]. We obtain

$$\alpha \geq 0.07 \frac{1}{(\upsilon r)^{5/3}} \frac{1}{\cos^2 \theta} . \quad (20)$$

We see that the stochasticity threshold is a rapidly decreasing function of υ and r. Therefore chaotic behaviour will appear, no matter how small the amplitude of the gravitational wave might be, provided that the initial momentum of the particles is sufficiently large. It is clear that in this case the high order principal resonances will overlap. On the other hand, if I_1 is small the chaotic behaviour will appear only if the wave amplitude is quite large and the relative frequency small, leading to the overlapping of the low order resonances.

Following the above argument, in Figure 1 we give the stochasticity threshold, α_{thr}, as a function of the low order resonance m, for different values of the wave propagation angle θ and $\upsilon = 1.8$. Notice that $\alpha_{thr} < 0.2$ and decreases rapidly as the order of resonance m increases.

3. THE FPK APPROACH

3.1. Analytic results

Following the FPK approach [9], a diffusion equation for the energy distribution function of particles averaged over the phases, $\mathcal{F}(I_1, t)$, can be written for the system described by the Hamiltonian (13),

$$\frac{\partial \mathcal{F}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial I_1}\left(D(I_1)\frac{\partial \mathcal{F}}{\partial I_1}\right). \tag{21}$$

To lowest order in α, eq. (21) describes a diffusion process in the variable $I_1 = p_x^2/2$ at constant I_3. The actual expression for the diffusion coefficient $D(I_1)$ depends on the assumptions for the phase dynamics [9,17]. In the random phase approximation, it reduces to the quasilinear result [18,19]

$$D(I_1) = \pi\alpha^2 \sum_m m^2 H_1^2 \qquad \delta\left(\frac{d\vartheta^{1*}}{d\lambda}\right) \tag{22}$$

which in our case reads

$$D(I_1) = \pi\alpha^2 \sum_m m^2 H_1^2 \qquad \delta(m^2 - \upsilon^2 \sin^2\theta\,[1 + 2I_1]). \tag{23}$$

Around each principal resonance of order m we may associate an *effective diffusion coefficient*, D_m, by averaging $D(I_1)$ over the region between two successive first order resonances:

$$D_m \equiv \langle D(I_1)\rangle = \frac{1}{\Delta m}\int_m^{m+1} D(I_1)dm. \tag{24}$$

To calculate D_m we use the facts that $\Delta m = 1$ and

$$\delta(f(m)) = \frac{\delta(m - m_0)}{|f'(m_0)|}, \tag{25}$$

where m_0 is a simple zero of $f(m)$ which, in this case, is given by eq. (15) [20]. Accordingly, we obtain

$$D_m \equiv \langle D(I_{1m}) \rangle = \frac{1}{2}\pi\alpha^2 m H_{1m}^2 , \tag{26}$$

where I_{1m} is the value of I_1 at each principal resonance of order m, which is found from eq. (15) to be of the form

$$I_{1m} = \frac{1}{2}\left(\frac{m^2}{\upsilon^2 \sin^2\theta} - 1\right) \tag{27}$$

and H_{1m} corresponds to the perturbation term of the Hamiltonian (13) for $I_1 = I_{1m}$. Equation (26), in terms of I_{1m}, reads

$$D_m = \tfrac{1}{2}\pi\alpha^2 \upsilon \sin\theta (1 + 2I_{1m})^{1/2} \times \\ \times [I_{1m}(1 + \cos^2\theta)J_m'' + (\tfrac{3}{2} + 4I_{1m})\cos^2\theta J_m]^2 . \tag{28}$$

We use the above relation in order to determine the analytical values of the diffusion coefficient, as it holds for any value of the parameters. It is clear that the diffusion coefficient scales with the wave amplitude α and, through the value of I_{1m}, with the order of resonance m. The diffusion coefficient reaches high values at low order resonances (small I_{1m}) when the wave amplitude is large. In the opposite case (small α) the diffusion becomes effective in the range of high order resonances and, thus, in large I_1. In both cases for a given α the diffusion increases as the action increases.

3.2. Numerical results

For the sake of numerical simplicity and in order to speed up numerical integration we investigate the case of low order resonances using $\alpha = 0.2$ and $\upsilon = 1.8$ throughout the whole of our numerical calculations. Since the results scale with the amplitude of the wave, α, the diffusion coefficient calculated is also expected to describe, at least qualitatively, the diffusive acceleration at more realistic values of α.

In order to verify that diffusion of particles due to their interaction with the gravitational wave does occur, we follow the orbits of a particle distribution on a surface of section, defined as the surface $\upsilon x^0 = 2n\pi$. In Figure 2 the π_1 versus x^1 plot and the time variation of the action I_1 of

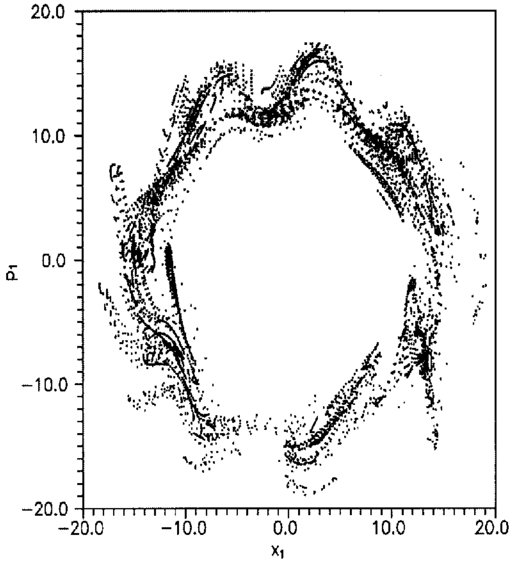

Figure 2a.

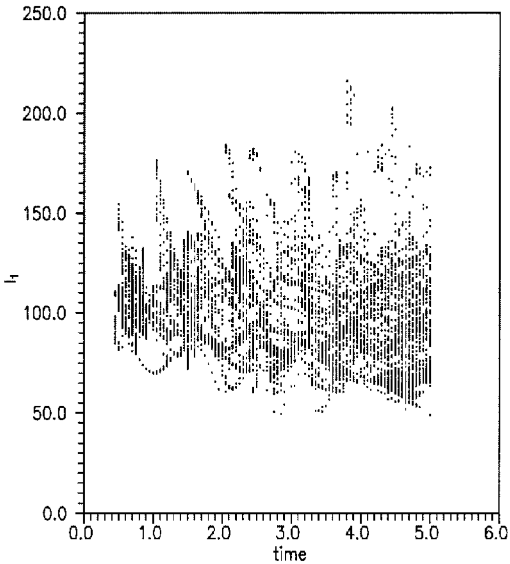

Figure 2b.

Figure 2. Surface of section plots for the case $\alpha = 0.2$, $\upsilon = 1.8$, $\theta = 20^o$. $N = 1000$ orbits with initial $I_1 = 106.5$ are presented: (a). The π_1 versus x^1 plot. (b). The time variation of the action I_1 of the distribution of orbits.

the distribution of orbits ($N = 1000$ with initial $I_1 = 106.5$) are presented for the case $\theta = 20°$.

Notice the distortion of the principal resonance (of order $m = 9$) due to the overlapping of the secondary resonances. This is due to the fact that, for the parameters used, the wave amplitude is large. Thus the overlapping occurs in a small time-scale. The diffusion in energy, in this case, is verified from the considerable spread around the initial action value $I_1 = 106.5$.

The numerical estimation of the diffusion coefficient is based on the integration of the Hamilton's equations of motion for a number of particles ($N = 1000$), having the same initial action I_1 and uniform angle distribution. The local diffusion coefficient is related to the average variations of the action I_1 through the expression [18,19]

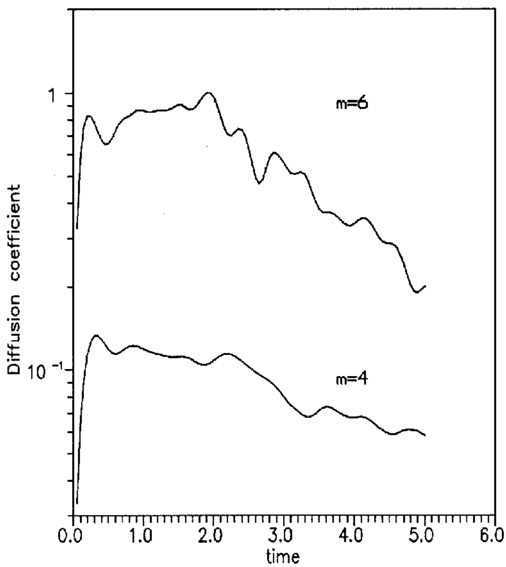

Figure 3. The numerical estimated local diffusion coefficients for $I_1 = 3.0$ and 7.0, corresponding to $m = 4$ and 6 respectively, for $\theta = 60°$ and $\upsilon = 1.8$.

$$D(I_1) \simeq \frac{\langle(\Delta I_1)^2\rangle - 2(\langle\Delta I_1\rangle)^2}{t}, \tag{29}$$

where

$$\langle\Delta I_1\rangle = \sum_{j=1}^{N} \frac{I_{1j}(t) - I_{1j}(0)}{N} \tag{30}$$

and

$$\langle (\Delta I_1)^2 \rangle = \sum_{j=1}^{N} \frac{[I_{1j}(t) - I_{1j}(0)]^2}{N} . \tag{31}$$

We have performed a number of computational runs, varying the initial value of I_1 ($2 \le I_1 \le 128$) and the propagation angles ($20^\circ \le \theta \le 60^\circ$). In Figure 3 the numerical estimated local diffusion coefficients for $I_1 = 3.0$ and 7.0, corresponding to $m = 4$ and 6 respectively, and for $\theta = 60^\circ$ are presented. Notice that the integration time is short, as for longer times diffusion over a large number of harmonics dominates, causing strong variations to the estimation of D. The plateau value of D is chosen as the diffusion coefficient for the above actions.

In Figure 4 the analytical and the numerical effective diffusion coefficient, as a function of the action I_1, for different angles θ is shown. Notice that the diffusion coefficient depends strongly on the propagation angle. There exists a good agreement between the numerically and analytically estimated values, indicating a power law dependence of the diffusion coefficient upon the action, of the general form

$$D(I_1) \simeq d_0 \, I_1^k , \tag{32}$$

where the values (analytically and numerically estimated) of the constant d_0 and the index k, with respect to the propagation angle θ are given in Table 1. The relative error between the analytically and numerically estimated values of the index k varies from 7% to 24%.

Table I. The analytically and numerically estimated values of d_0 and the index k with respect to the propagation angle θ.

θ	d_0		k	
	analytical	numerical	analytical	numerical
20°	0.095	0.037	2.254	2.455
35°	0.076	0.091	2.223	2.153
45°	0.052	0.025	2.160	2.407
60°	0.017	0.020	2.037	1.980

4. SOLUTION TO THE DIFFUSION EQUATION

We can easily solve the diffusion equation for the static case, i.e. $\partial_t \mathcal{F} = 0$. Then eq. (21) becomes

$$D(I_1) \frac{d^2 \mathcal{F}}{dI_1^2} + \frac{dD(I_1)}{dI_1} \frac{d\mathcal{F}}{dI_1} = 0 . \tag{33}$$

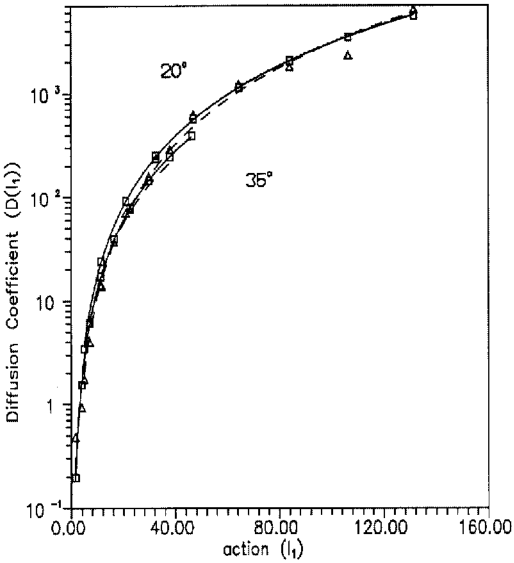

Figure 4a.

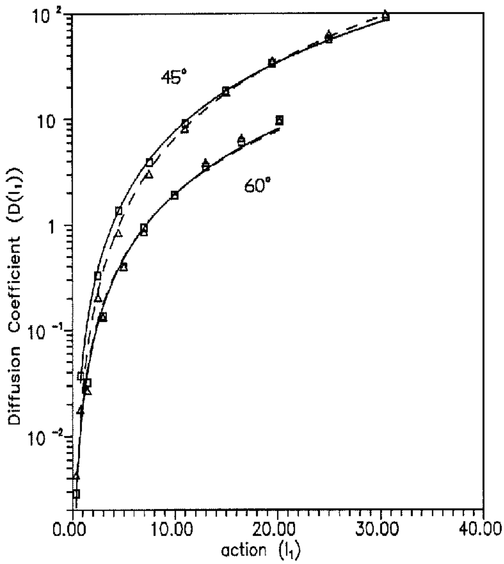

Figure 4b.

Figure 4. The diffusion coefficient $D(I_1)$, as a function of the perpendicular energy of the charged particle, I_1 for different angles of propagation, with $\alpha = 0.2$ and $\upsilon = 1.8$. The □ are analytical values and the △ are numerical ones: (a). For $\theta = 20^o$ and 35°. (b). For $\theta = 45^\circ$ and 60°.

We substitute the diffusion coefficient from eq. (32) to find the solution in a power law form

$$\mathcal{F}(I_1) = \frac{\mathcal{F}_o}{1-k} I_1^{-(k-1)} \tag{34}$$

with $\mathcal{F}_o$ constant. We must emphasize that the above solution is valid for relatively small values of I_1. In the case of very large energies ($I_1 \gg 1$) an analytic solution of the diffusion equation can be found, by considering the asymptotic form of the effective diffusion coefficient in the large energies approximation. For $I_1 \gg 1$, we have $r \longrightarrow \infty$ and therefore $m \approx \xi = \upsilon r$. Then the perturbation term of the Hamiltonian (13) reads

$$H_{R1} = -I_1[(1+\cos^2\theta)J''_m(m) + 4\cos^2\theta J_m(m)]. \tag{35}$$

In this case, the Bessel equation becomes

$$J''_m(m) \simeq -\frac{J'_m(m)}{m}, \tag{36}$$

and the asymptotic expansions

$$J_m(m) \sim 0.45 m^{-1/3}, \qquad J'_m(m) \sim 0.41 m^{-2/3}, \tag{37}$$

hold [21]. Therefore, in the large energies approximation, the effective diffusion coefficient reads

$$D_m(I_1) = A I_1^{5/6} - B I_1^{9/6} + C I_1^{13/6}, \tag{38}$$

where

$$\begin{aligned} A &= 2^{1/2}\pi\alpha^2\upsilon\sin\theta a^2, \\ B &= 2^{3/2}\pi\alpha^2\upsilon\sin\theta ab, \\ C &= 2^{1/2}\alpha^2\upsilon\sin\theta b^2, \end{aligned} \tag{39}$$

and

$$a = \frac{0.41}{2^{5/6}}\frac{1}{\upsilon^{5/3}}\frac{1+\cos^2\theta}{\sin^{5/3}\theta}, \qquad b = \frac{1.8}{2^{1/6}}\frac{1}{\upsilon^{1/3}}\frac{\cos^2\theta}{\sin^{1/3}\theta}. \tag{40}$$

Accordingly, the diffusion equation, in the static case, reads

$$\mathcal{F} = c\int \frac{dI_1}{A I_1^{5/6} - B I_1^{9/6} + C I_1^{13/6}}, \tag{41}$$

where c is an integration constant. Evaluation of the integral on the r.h.s of eq. (41) is possible only when $I_1 \neq (a/b)^{3/2}$, for which the energy distribution function appears a simple pole of order 2 [22]. In this case, we obtain

$$\mathcal{F} = \frac{6c}{C} b^2 \left[\frac{I_1^{1/6}}{(a - bI_1^{2/3})} + 3\frac{d}{4a} \left(\ln \frac{I_1^{1/6} + d}{I_1^{1/6} - d} + 2 \tan^{-1} \frac{I_1^{1/6}}{d} \right) \right], \tag{42}$$

where $d = (a/b)^{1/4}$. This result is simplified considerably in the perpendicular propagation case, i.e. $\theta = \pi/2$, for which eq. (41) gives

$$\mathcal{F} = \frac{6c}{A} I_1^{1/6}. \tag{43}$$

5. DISCUSSION AND CONCLUSIONS

We have studied the interaction of a charged particle, with a plane polarized gravitational wave propagating obliquely ($20° \leq \theta \leq 60°$) with respect to the direction of the ambient uniform magnetic field.

On the basis of Hamiltonian pertubation theory, previous work on this problem shows that the motion of the particles becomes chaotic [8]. Following this, we have derived analytical expressions for the stochasticity criterion, thus determining where, in phase space, resonance overlapping occurs, without any assumption regarding the values of the action and the propagation angle of the wave.

We have verified that diffusion of the particles in action I_1 occurs and we have applied the FPK approach, in order to derive analytical general expressions for the effective diffusion coefficient. Numerical integration of the exact equations of motion for particle distributions with the same initial action I_1 was also performed for the numerical estimation of the diffusion coefficient.

Both methods (analytical and numerical) revealed a power law dependence of the diffusion coefficient upon the action I_1 giving similar results, with small variations, on the power law index. Based on these results a steady state solution of the Fokker–Planck diffusion equation was found.

The diffusion coefficient scales with the wave amplitude α and the order of resonance m (and/ or through the resonance condition with the action I_1). For small α the diffusion is effective in high order resonances and thus in sufficient large actions. Diffusion of particles is present in low order resonances (small values of the action) only when the wave amplitude is large. In both cases the diffusion is increasing when the action is increasing.

There is also a strong relation between the diffusion coefficient and the propagation angle. As the angle decreases, the diffusion coefficient increases. This is due to the fact that the lower the angle, the greater is the amplitude of the wave for which stochastic motion occurs, leading to the fact that more resonances can overlap.

In conclusion, we believe that the FPK approach may describe to a good approximation the interaction of charged particles with a gravitational wave in the framework of the weak field theory, where the gravitational wave is just a small pertubation in a flat spacetime. It is clear that more work has to be done in the realistic case of a curved spacetime and in the full non-linear theory.

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