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Solar flare cellular automata interpreted as discretized MHD equations

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Abstract. We show that the Cellular Automaton (CA) model for Solar flares of Lu and Hamilton (1991) can be understood as the solution to a particular partial differential equation (PDE), which describes diffusion in a localized region in space if a certain instability threshold is met, together with a slowly acting source term. This equation is then compared to the induction equation of MHD, the equation which governs the energy release process in solar flares. The similarities and differences are discussed. We make some suggestions how improved Cellular Automaton models might be constructed on the basis of MHD, and how physical units can be introduced in the existing respective Cellular Automaton models. The introduced formalism of recovering equations from Cellular Automata models is rather general and can be applied to other situations as well.

Key words: chaos – MHD – methods: statistical – Sun: flares

1. Introduction

Cellular Automata (CA) models have been used to model solar flares (Lu and Hamilton 1991, hereafter LH; Lu et al. 1993; Vlahos et al. 1995; Georgoulis and Vlahos 1996), and they have been successful in reproducing several statistical properties of the latter, such as peak-flux distributions, total-flux distributions and duration distributions, as derived from HXR observations (Dennis 1985; Dennis 1988; Vilmer 1993). CA mimic the temporal evolution of the magnetic field on a spatial grid. They usually have a random loading function during their quiet evolution phase (build-up of magnetic field in the active region), changing however to a bursting type of evolution if a certain local threshold-criterion is fulfilled. The magnetic fields are then locally relaxed, loosely modeling an assumed magnetic reconnection process in this way. Chain reactions (avalanches) of such elementary bursts are interpreted as energy release events in flares.

In general, CA have been developed to model complex systems, i.e. systems which consist of a large number of interacting subsystems. The essence of the CA approach to such systems is to assume that the global dynamics, if described statistically, are not sensitive to the details of the elementary processes, the system has the property that most local information gets lost if viewed globally. The 'classical' approach to complex systems, on the other hand, is analytical: from a precise description of the elementary processes — in the optimum case involving the fundamental laws of physics, i.e. differential equations — one tries to understand a process globally. Both approaches have drawbacks and advantages. The CA approach does not explain what happens locally or over short time intervals, but it allows to understand the statistics of the global behaviour. The analytical approach may reveal insights into the local processes, but coupling this understanding to a global description is practically not feasible, mainly due to the large number of (in astrophysics even unobserved) boundary conditions. In this sense, the two approaches can be considered as complementary, and a description of a complex system should ideally combine them.

Concerning the problem of solar flares, such a combination is still missing. In this article, we try to make a first step towards this direction, starting from the point where the two approaches touch, namely at the scale where the local micro-physics can be summarized into simple CA evolution rules.

The analytical (micro-physical) theory of the processes in solar active regions is kinetic plasma physics, or, with some idealizations, MHD, i.e. a set of partial differential equations. The task is to establish a connection between the solar flare CA rules and the (local) MHD equations. The way we choose to do so is first to see how the CA rules are related to differential equations, and then to compare these equations to the relevant ones of MHD. We note that Lu (1995) made a general discussion of whether there exist continuous driven dissipative systems which show analogous features as CA models for solar flares, namely avalanches. He found a general type of such continuous systems, however, they were not derived from a given CA model, but constructed in order to mimic the general statistical properties of CA dynamics. Our approach, on the other hand, starts from a particular (solar flare) CA and derives the continuous system which exactly corresponds to this CA, establishing thus a translation scheme to go from a given CA to a partial differential equation (and vice-versa), which in turn is very general.

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We will concentrate this inquiry on the model of LH, since it was the first CA model suggested for an application to flares, and the later developed CA models, though being improved in details, still have the essential features of this first model.

We will first review the CA model of LH (Sect. 2), the starting point of our discussion. In Sect. 3 we will recover the differential equation behind this CA, giving the derivation in enough details so that the way of proceeding may be applied to other CA models. The result, the explicit form of this equation, will be stated in Sect. 4. In Sect. 5, we discuss the relation of this equation to the MHD equations relevant to solar flares, unveiling thus the nature of the process, the assumptions, and the simplifications which are hidden behind the CA model of LH. In the conclusion (Sect. 6) we finally make some suggestions on how the results of the existing CA models can be interpreted more thoroughly by introducing physical units, and how improved (in the sense of more physically motivated) CA models could be constructed (in a parallel paper the idea of deriving CA rules from MHD will be pursued (Vassiliadis et al. 1998, as a continuation of Vassiliadis et al. 1996)).

2. Review of the CA model of Lu and Hamilton

The vector of the magnetic field $\boldsymbol{b}_{i,j,k}$ is given on a 3-dimensional cubic grid. After assigning random initial values to the magnetic field at each grid site, the evolution is governed by two laws: In the quiet phases the magnetic field evolves according to

$$\boldsymbol{b}_{i,j,k}(t+1) = \boldsymbol{b}_{i,j,k}(t) + \boldsymbol{s}_{i,j,k}(t) \tag{1}$$

where $\boldsymbol{s}_{i,j,k}(t)$ is an asymmetrically valued random source-term acting on a characteristic time scale which is large compared to the one of the instability that will be described in the following. The quantity

$$d\boldsymbol{b}_{i,j,k} := \boldsymbol{b}_{i,j,k} - \frac{1}{6}\sum_{n.n.} \boldsymbol{b}_{n.n.} \tag{2}$$

is used in an instability criterion ($n.n.$ labels the 6 nearest neighbours in the 3D cubic grid): If

$$|d\boldsymbol{b}_{i,j,k}| > b_c \tag{3}$$

at position i,j,k (for some given threshold b_c), the evolution changes into a fast burst mode: the source term is not acting anymore, and the magnetic field evolves as

$$\boldsymbol{b}_{i,j,k}(t+1) = \boldsymbol{b}_{i,j,k}(t) - \frac{6}{7} d\boldsymbol{b}_{i,j,k} \tag{4}$$

for the point where the instability occurs, and

$$\boldsymbol{b}_{n.n.}(t+1) = \boldsymbol{b}_{n.n.}(t) + \frac{1}{7} d\boldsymbol{b}_{i,j,k} \tag{5}$$

for its six nearest neighbours. Inserting these two equations into the definition of $d\boldsymbol{b}_{i,j,k}$ (Eq. 2), it turns out that the latter quantity vanishes after one time step:

$$d\boldsymbol{b}_{i,j,k}(t+1) = 0 \tag{6}$$

The energy released during one such burst-event is assumed to be

$$e_R(t) = \frac{6}{7}|d\boldsymbol{b}_{i,j,k}|^2 \tag{7}$$

If all instabilities in the grid have been relaxed, then the evolution is again in the slow mode (Eq. 1).

The CA model of LH was the first one applied in the context of solar flares. Later developed CA models basically use the same set-up, just changing slightly the above rules. In the following, we concentrate on the LH model, but we emphasize that our approach can be applied to any CA model developed so far for solar flares.

3. Recovering the differential equation behind the CA of LH

3.1. *Quiet evolution; the instability criterion; the unstable points and their nearest neighbours*

The expressions used in LH involve differences of the magnetic fields in space and time, and it is natural to interpret the difference-expressions as discretized differential-expressions. We start with considering the control quantity $d\boldsymbol{b}_{i,j,k}$ (Eq. 2), considering the x-coordinate only:

$$\begin{aligned} db^x_i &= b^x_{i,j,k} - \tfrac{1}{6}\sum b^x_{n.n.} = \tfrac{1}{6}\Big(\sum \big(b^x_{i,j,k} - b^x_{n.n.}\big)\Big) \\ &= \tfrac{1}{6}\Big(\big(b^x_{i,j,k} - b^x_{i+1,j,k}\big) + \big(b^x_{i,j,k} - b^x_{i-1,j,k}\big) \\ &\quad + \big(b^x_{i,j,k} - b^x_{i,j+1,k}\big) + \big(b^x_{i,j,k} - b^x_{i,j-1,k}\big) \\ &\quad + \big(b^x_{i,j,k} - b^x_{i,j,k+1}\big) + \big(b^x_{i,j,k} - b^x_{i,j,k-1}\big)\Big) \end{aligned} \tag{8}$$

We now explicitly introduce Δh, the distance between adjacent grid sites (LH assume $\Delta h = 1$), and multiply the equation with $1/\Delta h^2$, whence

$$\begin{aligned} \frac{db^x_i}{\Delta h^2} &= -\tfrac{1}{6}\,\tfrac{1}{\Delta h^2}\Big(\big(b^x_{i+1,j,k} - 2\,b^x_{i,j,k} + b^x_{i-1,j,k}\big) \\ &\quad + \big(b^x_{i,j+1,k} - 2\,b^x_{i,j,k} + b^x_{i,j-1,k}\big) \\ &\quad + \big(b^x_{i,j,k+1} - 2\,b^x_{i,j,k} + b^x_{i,j,k-1}\big)\Big) \end{aligned} \tag{9}$$

The r.h.s. obviously is — besides a factor — the discretization of $\frac{\partial^2 b_x(\boldsymbol{x},t)}{\partial x^2}$, done in a standard way (see e.g. Gerald and Wheatley 1989). Treating the other coordinates analogously, we can identify

$$\lim_{\Delta h \to 0} \frac{d\boldsymbol{b}_{i,j,k}}{\Delta h^2} = -\frac{1}{6}\nabla^2 \boldsymbol{b}(\boldsymbol{x},t) \tag{10}$$

The time evolution of the process is the following: In the non-critical state, the evolution is, according to Eq. (1),

$$\boldsymbol{b}_{i,j,k}(t+1) - \boldsymbol{b}_{i,j,k}(t) = \boldsymbol{s}_{i,j,k}(t) \tag{11}$$

or, introducing the time step Δt, which was assumed to be always 1 in LH, we multiply with $1/\Delta t$

$$\frac{\boldsymbol{b}_{i,j,k}(t+\Delta t) - \boldsymbol{b}_{i,j,k}(t)}{\Delta t} = \frac{\boldsymbol{s}_{i,j,k}(t)}{\Delta t} \tag{12}$$

so that for $\Delta t \to 0$, we find

$$\frac{\partial \boldsymbol{b}(\boldsymbol{x},t)}{\partial t} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \boldsymbol{s}(\boldsymbol{x},t) \tag{13}$$

The singularity on the r.h.s. is removed by interpreting $\boldsymbol{s}(\boldsymbol{x},t)$ as $\boldsymbol{S}(\boldsymbol{x},t)\,\Delta t$, where $\boldsymbol{S}(\boldsymbol{x},t)$ is the field injected per unit time, and saying that Lu and Hamilton consider the field accumulated over the time step Δt. We get finally the quiet evolution

$$\frac{\partial \boldsymbol{b}(\boldsymbol{x},t)}{\partial t} = \boldsymbol{S}(\boldsymbol{x},t) \tag{14}$$

The instability criterion (Eq. 3) turns into

$$\frac{1}{6}\left|\nabla^2 \boldsymbol{b}(\boldsymbol{x},t)\right| > \lim_{\Delta h \to 0} \frac{b_c}{\Delta h^2} \tag{15}$$

(by first multiplying Eq. (3) with $1/\Delta h^2$, and then inserting Eq. 10). Again, we have to interpret b_c as a 'cumulative threshold', in the sense that $b_c = B_c\,\Delta h^2$, so that the criterion is

$$\frac{1}{6}\left|\nabla^2 \boldsymbol{b}(\boldsymbol{x},t)\right| > B_c \tag{16}$$

If the instability criterion is met, then according to Eq. (4) we have for the point where the instability occurs, inserting Eq. (10) for $d\boldsymbol{b}_{i,j,k}$,

$$\boldsymbol{b}_{i,j,k}(t+1) - \boldsymbol{b}_{i,j,k}(t) = -\frac{6}{7}\left(-\frac{1}{6}\right)\Delta h^2\,\nabla^2 \boldsymbol{b}(\boldsymbol{x},t) \tag{17}$$

or by dividing by the time-step Δt

$$\frac{\boldsymbol{b}_{i,j,k}(t+\Delta t) - \boldsymbol{b}_{i,j,k}(t)}{\Delta t} = \frac{1}{7}\left(\frac{\Delta h^2}{\Delta t}\right)\nabla^2 \boldsymbol{b}(\boldsymbol{x},t) \tag{18}$$

and for $\Delta t \to 0$, $\Delta h \to 0$, we find

$$\frac{\partial \boldsymbol{b}(\boldsymbol{x},t)}{\partial t} = \lim_{\substack{\Delta t \to 0 \\ \Delta h \to 0}} \left(\bar{\eta}\frac{\Delta h^2}{\Delta t}\right)\nabla^2 \boldsymbol{b}(\boldsymbol{x},t) \tag{19}$$

where we have introduced the diffusion constant $\bar{\eta}$. Again, a singularity seems to appear on the r.h.s. We can absorb it by assuming $\bar{\eta} = \eta\,\Delta t/\Delta h^2$ for some η, i.e. we interpret Lu and Hamilton as having used a 'cumulative' diffusion coefficient $\bar{\eta}$, which, for $\Delta t = \Delta h = 1$, adopts the value $1/7$. Whence we have

$$\frac{\partial \boldsymbol{b}(\boldsymbol{x},t)}{\partial t} = \eta \nabla^2 \boldsymbol{b}(\boldsymbol{x},t) \tag{20}$$

with the diffusion coefficient

$$\eta = \frac{1}{7} \tag{21}$$

It remains to consider the nearest neighbours of the points where the instability criterion is fulfilled. They evolve according to a different rule than the center point, and we have to check whether the different rules are compatible. The nearest neighbours evolve according to Eq. (5), so that, by using Eq. (10),

$$\boldsymbol{b}_{i,j,k}(t+1) - \boldsymbol{b}_{i,j,k}(t) = \frac{1}{7}\left(-\frac{1}{6}\right)\Delta h^2 \nabla^2 \boldsymbol{b}(\boldsymbol{x},t) \tag{22}$$

or again

$$\frac{\boldsymbol{b}_{i,j,k}(t+\Delta t) - \boldsymbol{b}_{i,j,k}(t)}{\Delta t} = -\frac{1}{42}\left(\frac{\Delta h^2}{\Delta t}\right)\nabla^2 \boldsymbol{b}(\boldsymbol{x},t) \tag{23}$$

and for $\Delta t \to 0$, $\Delta h \to 0$, we find

$$\frac{\partial \boldsymbol{b}(\boldsymbol{x},t)}{\partial t} = \lim_{\substack{\Delta t \to 0 \\ \Delta h \to 0}} \left(\bar{\eta}'\frac{\Delta h^2}{\Delta t}\right)\nabla^2 \boldsymbol{b}(\boldsymbol{x},t) \tag{24}$$

with a dummy diffusion constant $\bar{\eta}'$, or absorbing again the seeming singularity by introducing η' so that $\bar{\eta}' = \eta'\frac{\Delta t}{\Delta h^2}$,

$$\frac{\partial \boldsymbol{b}(\boldsymbol{x},t)}{\partial t} = \eta' \nabla^2 \boldsymbol{b}(\boldsymbol{x},t) \tag{25}$$

with the diffusion coefficient

$$\eta' = -\frac{1}{42} \tag{26}$$

3.2. Problems and inconsistencies

A. Continuity

In what we have recovered so far, a site which becomes unstable has a temporal evolution (Eqs. (20) and (21)) which is different from the one of the nearest neighbour sites (Eqs. (25) and (26)): the diffusion coefficients are different. If we consider the evolution law on the grid as a discretized partial differential equation (PDE), then we may choose the grid size arbitrarily. If we let it go to zero, then the central point and its nearest neighbours approach each other until they coincide. For physical reasons, we must demand continuity of the fields, which can be achieved only if the coefficients in the PDE are continuous, too, i.e. the evolution laws for the central point and its nearest neighbours must coincide for $\Delta h \to 0$. However, from Eqs. (21) and (26) it is clear that this condition is not satisfied, the diffusion coefficients are different.

The way out of this dilemma is realizing that the grid size Δh in the CA of LH is not an arbitrary quantity, but it is physically meaningful, it is a property of the considered system, some kind of a characteristic size (termed l_0 in the following). Whence, the preliminary interpretation we give so far to the CA dynamics in the LH model is that, in the burst mode, the system evolves according to a diffusion equation (Eq. 20), with a diffusivity η which is given by Eq. (21) at the point where the instability criterion is met, drops to a value given by Eq. (26) for points which are one characteristic length l_0 away from the central point, and finally settles to 0 for points which are further than two characteristic lengths away (in Fig. 1 a sketch of η vs. x (1D) is given).

B. Two physical and a mathematical problem

The interpretation given so far is still not satisfying, for physical as well as mathematical reasons: (i) The diffusivity η' in the neighbourhood of an unstable point is negative (Eq. 26).

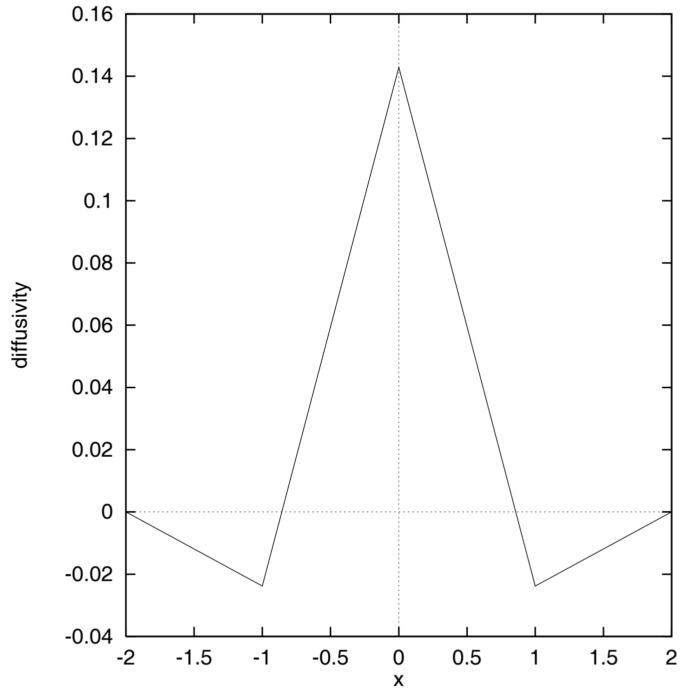

Fig. 1. 1D-sketch of the spatial behaviour of the diffusivity. The unstable site is located at $x = 0$, and the characteristic size (radius) l_0 of the affected volume is assumed to be 1.

This cannot be motivated with reasonable physical arguments, and mathematically, such a diffusion equation has exploding solutions. (ii) Both diffusion coefficients, the central and the peripheral one, carry still an element which is reminiscent of the particular grid chosen, namely the numerical factors 1/7 in Eq. (21) and 1/42 in Eq. (26). Following the argumentation in LH, these factors in a different grid would have to be $1/(n+1)$ in Eq. (21) and $1/(n(n+1))$ in Eq. (26), where n is the number of nearest neighbours (e.g. in a grid with hexagonal prismas one has $n = 8$). Obviously, the particular n appears still in the continuous formulation of the process, which is contradictory. Letting $n \to \infty$ gives useless results for the central point (Eq. 21): if we replace the 7 in Eq. (21) with $n+1$ and try to absorb the singularity which appears for $n \to \infty$, then, since we already identified Δh with l_0, we would have to introduce $\Delta t = \tau/n$, for some given τ (any characteristic time of the process). Therewith, however, the time-step in the CA would depend on the number of nearest neighbours, which is in contradiction with all the CA models. (iii) Having the diffusivity varying from a positive value to a negative value is like having two different processes acting, but the two processes are actually connected: the neighbourhood of an unstable point receives exactly the field which is lost by the central point.

3.3. The consistent approach

The conclusion so far is that the derived PDEs (Eqs. 20 and 25) cannot be the continuous version of the CA in its bursting phase. To find the correct continuous formulation for the CA of LH, one has to reconsider the CA rules (Sect. 2) for the case where an instability occurs: The central point of the unstable region evolves according to Eq. (4), and the neighbour-sites according to Eq. (5). This evolution is characterized by the fact that after a time step Δt, we have $d\boldsymbol{b}_{i,j,k}(t+\Delta t) = 0$ (Eq. 6): the site is no more unstable, the fields are flattened, or, in continuous language, since $\lim_{\Delta h \to 0} d\boldsymbol{b}_{i,j,k}/\Delta h^2 = -1/6\,\nabla^2 \boldsymbol{b}$ (Eq. 10), we have $\nabla^2 \boldsymbol{b}(t+\Delta t) = 0$.

Every burst in the CA is such that after a time step Δt the magnetic field has diffused through a distance Δh and is completely relaxed. This implies that Δt has also a physical meaning, it is a relaxation time τ_d of the ongoing process, and Δh is the characteristic size l_0 of the system, as also stated above. This means that the discrete evolution laws (Eqs. 4 and 5) should not be interpreted as the equivalent to differential equations (as done in Eqs. 20 and 25), though this is formally possible, but they describe the **solution** to a continuous equation which has to be found from the following general properties of the ongoing process:

1. After the relaxation time τ_d, the magnetic field around an unstable site is almost homogeneously distributed over the sphere S_{l_0} with radius the characteristic-size of the system. Thereby, homogeneity is measured through $\nabla^2 \boldsymbol{b}$:
$$\nabla^2 \boldsymbol{b}(\boldsymbol{x}, t) = 0 \quad \text{after } \tau_d, \text{ for } \boldsymbol{x} \in S_{l_0} \tag{27}$$
2. The process is not interacting with the region outside the sphere S_{l_0} around the unstable point, the field in this sphere is conserved, neither in- or outflow of field takes place:
$$\frac{\partial}{\partial t} \int\limits_{S_{l_0}} \boldsymbol{b}(\boldsymbol{x}, t)\, dV = 0 \tag{28}$$

We are looking for a differential equation which fulfills these two conditions: generally spoken, we need to determine $\boldsymbol{f}$ in $\partial \boldsymbol{b}/\partial t = \boldsymbol{f}(\boldsymbol{b})$ or $\partial^2 \boldsymbol{b}/\partial t^2 = \boldsymbol{f}(\boldsymbol{b})$ such that the above properties are recovered. Of course, there is an infinite number of solutions for $\boldsymbol{f}$, and we will look for the simplest one (higher order terms cannot be motivated by the few properties wanted). To find $\boldsymbol{f}$, we first note that the quantity which drives the process obviously is $\nabla^2 \boldsymbol{b}$, since if this quantity is not zero, then the process acts, until this Laplacian vanishes. Now assume that $\nabla^2 b_x$ is negative in some region (considering the first component of $\nabla^2 \boldsymbol{b}$, the others are analogous), then b_x is a convex function in this region, and it has to decrease in order that the convexity disappears (and thereby $\nabla^2 b_x$ goes to 0). Analogously, b_x has to increase where $\nabla^2 b_x$ is positive. Whence, if we assume that $\partial b_x/\partial t$ is proportional to $\nabla^2 b_x$ we have a simple equation with the wished property, and analogously for the other components, so that, in vector form,

$$\frac{\partial \boldsymbol{b}}{\partial t} = \eta \nabla^2 \boldsymbol{b} \tag{29}$$

is the wanted equation, with some constant η, which can be interpreted as a diffusivity, and for which we must have

$$\eta = \frac{l_0^2}{\tau_d} \tag{30}$$

in order that the system has relaxed in a volume S_{l_0} after a time τ_d.

The second wanted property (Eq. 28) yields the boundary condition: We have (for one of the components)

$$0 = \int_{S_{l_0}} \frac{\partial b_x}{\partial t}\, dV = \eta \int_{S_{l_0}} \nabla^2 b_x \, dV = \eta \int_{\partial S_{l_0}} (\boldsymbol{n}\nabla) b_x \, ds \qquad (31)$$

where we have first inserted the differential equation (Eq. 29) and then used Gauss' theorem ($\boldsymbol{n}$ is a unit vector normal to the surface ∂S_{l_0} of the sphere S_{l_0}). We therefore conclude that the boundary conditions are

$$(\boldsymbol{n}\nabla)\boldsymbol{b}(\boldsymbol{x}, t) = 0, \quad \text{for } \boldsymbol{x} \in \partial S_{l_0}, \qquad (32)$$

for all times t during which the burst process is acting. (From this derivation it is clear that the form of the boundary conditions which describe no interaction with the outside region depend on the differential equations and cannot be stated in general, irrespective of what equation should have this property.)

Going from the uncovered continuous equation (Eq. 29) back to the CA, it is now clear that running the CA of LH corresponds to solving the diffusion equation (Eqs. 29 with 32) in a particular way: time and space are discretized by using a grid-size which is the characteristic length l_0 and a time step which is the diffusion time τ_d of the system. If the sphere S_{l_0} contains n neighbour points in a discretization, then, since we know that after τ_d the fields in this sphere are flattened, we may just redistribute the fields by using a dilution factor $1/n$. Exactly this is implemented in the evolution rules for central point (Eq. 4), the nearest neighbours (Eq. 5), and in that grid sites further away are not influenced. In this sense, the CA solves the diffusion equation: its trivial solution (flattening of the magnetic field in the sphere S_{l_0} after time τ_d) is the basis of the evolution law of the central point and its nearest neighbours. We note, however, that these evolution rules are not unique, they just must fulfill that (in continuous language) $\nabla^2 \boldsymbol{b}$ is 0 after one time-step. An alternatively possible rule would for instance be the complete equi-distribution of the fields (see also the remark in the next section).

This way of proceeding simplifies greatly the solution of the diffusion equation (Eq. 29), it has the disadvantage, however, that nothing is known on the dynamics on time scales shorter than τ_d or length scales shorter than l_0. If this evolution would be of interest, then the CA frame would not help anymore, and a usual PDE integration scheme would have to be used. On the other hand, the advantage of this approach is that one can run a simplified model without implementing the unknown details of the process, and can monitor therewith the global evolution of a spatially extended complex system, given that it consists of many localized, randomly triggered diffusion events.

4. Result

We can summarize the result of the previous section in the claim that the CA of Lu and Hamilton (see Sect. 2) is equivalent to the following continuous system:

Given is a magnetic field $\boldsymbol{B}(\boldsymbol{x}, t)$ in 3D-space. The initial condition is a random distribution. The field evolves according to

$$\frac{\partial \boldsymbol{B}(\boldsymbol{x}, t)}{\partial t} = \eta \nabla^2 \boldsymbol{B}(\boldsymbol{x}, t) + \boldsymbol{S}(\boldsymbol{x}, t) \qquad (33)$$

with the diffusion coefficient

$$\eta(\boldsymbol{x}, t) = \begin{cases} \frac{l_0^2}{\tau_d} & (\boldsymbol{x}, t) \in \Big\{ (\boldsymbol{x}', t') \;\Big|\; \|\boldsymbol{x}' - \boldsymbol{x}_0\| \le l_0 \\ & \quad \text{and } 0 \le t' - t_0 \le \tau_d \text{ and } (\boldsymbol{x}_0, t_0) \in \\ & \quad \Big\{ (\boldsymbol{x}_0', t_0') \;\Big|\; \frac{1}{6} \left| \nabla^2 \boldsymbol{B}(\boldsymbol{x}_0', t_0') \right| > B_c \Big\} \Big\} \\ 0 & \text{else} \end{cases} \qquad (34)$$

and the boundary condition

$$(\boldsymbol{n}\nabla)\boldsymbol{B}(\boldsymbol{x}, t) = 0, \quad \text{for } \boldsymbol{x} \in \partial S_{l_0}(\boldsymbol{x}_0) \qquad (35)$$

where $S_{l_0}(\boldsymbol{x}_0)$ denotes the sphere of radius l_0 around $\boldsymbol{x}_0$, and $\partial S_{l_0}(\boldsymbol{x}_0)$ its surface. τ_d is the diffusive time and l_0 the characteristic length of the system. The source-function $\boldsymbol{S}(\boldsymbol{x}, t)$ is asymmetric in its values and random in space and time, with $|\boldsymbol{S}(\boldsymbol{x}, t)| < B_c$, a time scale much larger than τ_d, and spatial correlations which decay over a length smaller than l_0. An example of such a process is a Poisson process in time and space, with mean time-interval between two shots $>> \tau_d$ and mean spacing between two shots $>> l_0$.

Eqs. (33), (34) and (35) describe a system in which a field quantity is randomly increased, until it reaches a threshold. This turns on a fast diffusion process which takes place over a volume of size l_0 and acts during a time τ_d (it does not stop when the critical quantity falls below the threshold, but only when the critical quantity has reached a value 0). It is thus a localized, threshold-dependent, fast diffusion process, completely disconnected from the surrounding region. The restructuring of the magnetic field may eventually cause that in an adjacent volume the instability criterion is met, and so on, so that an avalanche-like event may occur. Lu (1995) has shown that continuous systems with localized diffusion events can show avalanche behaviour. The critical point is that in a CA the region which is unstable completes its diffusive process before the neighbour sites may eventually start to become unstable, whereas in continuous systems, neighbour regions may become unstable at any time instant during the primary instability, causing effects which are difficult to predict, in general.

In 1D, the diffusive part of the equation can be solved analytically. If an instability starts at $t = t_0$ and $x = x_0$ with an initial distribution of the fields $B_0(x)$ in $[a, b]$ (with $b - a = l_0$), then we have

$$B_x(x, t) = \frac{1}{2\sqrt{\pi}} \frac{1}{\sqrt{\eta(t - t_0)}} \int_{-\infty}^{\infty} e^{-\frac{(x - x')^2}{4\eta(t - t_0)}} \tilde{B}_0(x')\, dx' \qquad (36)$$

where $\tilde{B}_0(x) = B_0(x)$ in $[a, b]$, and outside $[a, b]$ it is defined in such a way that it is an even function with respect to $x = a$ and $x = b$ (which implies that $\tilde{B}_0(x)$ is a particular choice of a periodic continuation). As an illustration, we plot in Fig.

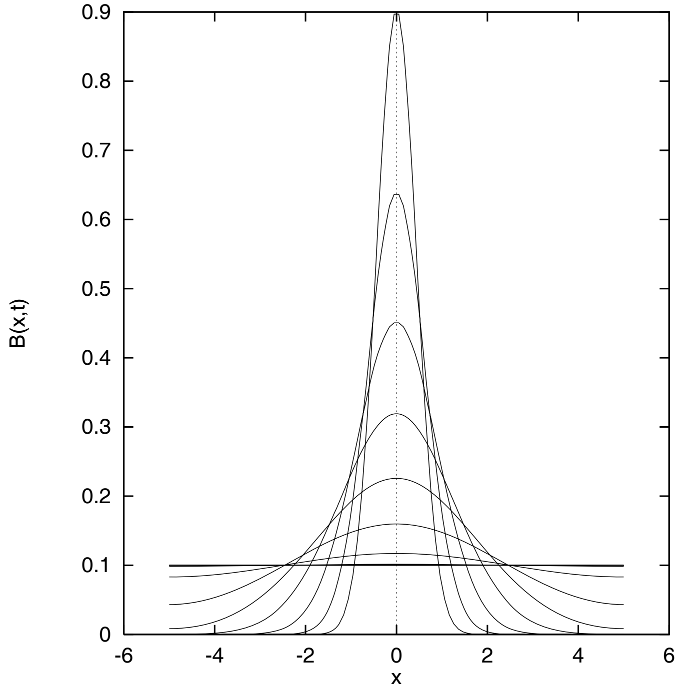

Fig. 2. Temporal evolution in 1D space of the magnetic field, $B(x,t)$, undergoing the diffusive process Eqs. (33), (34), (35). For details see text.

2 $B(x,t)$ for the times $t = 2^k\, 1000/1024$ ($k = 0, 1, ..., 10$), with the initial condition $B_0(x) = \delta(x)$, $a = -l_0/2$, $b = l_0/2$, $l_0 = 10$, $\eta = 0.1$, and consequently $\tau_d = 1000$: The field goes asymptotically to a flat state, which has effectively been reached already after the time τ_d ($k = 10$ in Fig. 2). We note that in 1D, a straight line with any slope would also be an asymptotic solution to Eq. (33). However, the boundary condition (Eq. 35) demands that the field has zero slope at the two edges, which introduces convex regions of $B(x,t)$, and which drive then the diffusion equation again until a flat distribution of the fields is reached.

Of course, the equation could also be solved numerically on a spatial grid. The essential point of the CA of LH is that the equation is not discretized to solve it, but directly its solution after the relaxation time is implemented in the form of CA evolution-rules — which is feasible since this solution can be expected to be trivial, namely a flat distribution of the fields after the diffusive time, as illustrated above through the analytical example. If one chooses a grid-size

$$\Delta h = l_0 \tag{37}$$

and a time-step

$$\Delta t = \tau_d \tag{38}$$

and makes the identifications $\boldsymbol{s}_{i,j,k}(t) = \boldsymbol{S}(\boldsymbol{x},t)\Delta t$ (for Eq. 1), and $b_c = B_c\, \Delta h^2$ (for Eq. 3), then the CA rules in Sect. 2, since they correspond to flattening of the field ($d\boldsymbol{b}$ is 0 after one time step), are a solution of the diffusion equation after the diffusive time has elapsed (leaving though some rest-fluctuations in the field, since the latter is only flattened and not exactly equidistributed, as we would demand from the analytical example given above).

Note that if one wants to have information on the process on time scales smaller than Δt or spatial scales smaller than Δh then the CA of LH is useless. One would have to construct a different CA, with different rules, which would be nothing else than an integration scheme of the PDE Eq. (33). But due to its simplicity, the CA allows to study the statistics of the large scale events, such as the possible occurrence of avalanches, and this is its true benefit.

5. Discussion: the context of solar flares

We have shown that the CA of LH (Sect. 2) can be interpreted as the solution to a diffusion equation plus a source term. Here now, we will give some examples to demonstrate the benefit gained through this alternative description: the somewhat neutral rules of the CA can be interpreted (or modified) on the basis of our understanding of MHD equations, as related to solar flares.

Generally, flares are considered to be made up by a large number of reconnection events distributed somehow over an active region (Parker 1988; Parker 1989). In MHD, the processes in the active region are described by the induction equation

$$\frac{\partial \boldsymbol{B}(\boldsymbol{x},t)}{\partial t} = \eta \nabla^2 \boldsymbol{B}(\boldsymbol{x},t) + \nabla \wedge (\boldsymbol{v} \wedge \boldsymbol{B}) \tag{39}$$

plus a momentum equation for the evolution of the velocity field $\boldsymbol{v}$ (the currents and the electric field can be considered as secondary quantities). In general then, the evolution of the magnetic field is governed by the convective term (2nd term on the r.h.s. of Eq. 39), since η is very small, mostly. Accidentally, this convective evolution may create small scale structures where $\eta = l_0^2/\tau_d$ is not small anymore, and the diffusive term dominates the evolution of the magnetic field (1st term on the r.h.s. of Eq. 39). This diffusive regime is characterized by its spatial scale l_0 and its temporal scale τ_d. Both scales are bigger than the respective ones of the current sheet and the reconnection process, they characterize the volume and the time in which the magnetic field has been reconnected and the free magnetic energy has been released (for details see Biskamp 1994, and references therein).

Having the described picture of the flare scenario in mind, we can interpret the CA of LH (Sect. 2), not by considering the CA rules, however, but by looking at the continuous version of the CA model (Sect. 4):

— The PDE corresponding to the CA of LH (Eq. 33) has two modes, the stable and the unstable one:
  — In the stable mode, Eq. (33) reduces to
$$\frac{\partial \boldsymbol{B}(\boldsymbol{x},t)}{\partial t} = \boldsymbol{S}(\boldsymbol{x},t) \tag{40}$$
  (see Eq. 34). Therewith, it mimics the induction equation (Eq. 39) in the convective regime (i.e. outside a reconnection region, where the diffusive term is negligible), describing in a simplifying way the convective term ($\nabla\wedge(\boldsymbol{v}\wedge\boldsymbol{B})$), which actually should reflect the turbulent motion in the active region and plasma inflows

from the photosphere, through a simple random function, neglecting thus completely any structures which would be due to organized fluid motions.
— If the Laplacian $\left|\nabla^2 \boldsymbol{B}(\boldsymbol{x}'_0, t)\right|$ of the magnetic field exceeds a certain threshold (Eq. 34), then Eq. (33) reduces to

$$\frac{\partial \boldsymbol{B}(\boldsymbol{x},t)}{\partial t} = \eta \nabla^2 \boldsymbol{B}(\boldsymbol{x},t) \tag{41}$$

The loading term $\boldsymbol{S}$ can be neglected since its time scale is much slower than the one of the diffusive process (which is one of the assumptions of the CA model). Eq. (41) corresponds to the induction equation in the diffusive regime, i.e. there where η is so large that the convective term can be neglected in Eq. (39). According to Eq. (34), this diffusion is bounded to a region of radius l_0 around the point where the instability criterion is met. Obviously, from the point of view of MHD, l_0 is the length scale of the diffusive region, which is naturally assumed to be bounded. LH assume this l_0 to be the same for every possibly occurring reconnection event, and moreover, they assume all these reconnection events to have the same diffusive time τ_d.

— The amount of released energy during one diffusion (reconnection) event is assumed by LH to be $1/42\, l_0^4 \left|\nabla \boldsymbol{B}\right|^2$ (Eqs. 7 and 10). This is a rough approximation, which can be put on more physical grounds: From the physical point of view, the released energy is the difference between the initial and the final magnetic energy: $E_R = \int_{l_0^3} \left(\boldsymbol{B}_{ini}^2 - \boldsymbol{B}_{fin}^2\right) dV/2\mu$, where the volume integral is over the diffusive region l_0^3. As stated in Sect. 4, $\boldsymbol{B}_{fin}$ is (more or less) flattened in the diffusive volume, i.e. $\boldsymbol{B}_{fin} = \int \boldsymbol{B}_{ini}\, dV / \int dV =< \boldsymbol{B}_{ini} >$, the spatial mean value of the initial field in l_0^3. Whence, we get for the released energy

$$\begin{aligned} E_R &= \int\limits_{l_0^3} \left(\boldsymbol{B}_{ini}^2 - < \boldsymbol{B}_{ini} >^2\right) dV/2\mu \\ &= \mathrm{var}_{l_0^3}[\boldsymbol{B}_{ini}]\, l_0^3/2\mu \end{aligned} \tag{42}$$

where we have identified the expression for the variance of the initial $\boldsymbol{B}$ field in l_0^3. In discrete language, this turns into

$$\begin{aligned} E_R &= \mathrm{var}_{j=1,\ldots,N}[\boldsymbol{B}_j]\, l_0^3\, N/2\mu \\ &= \sum_{k=1}^{N} \left(\boldsymbol{B}_k^2 - \frac{\left(\sum_{j=1}^N \boldsymbol{B}_j\right)^2}{N} \right) l_0^3\,/2\mu \end{aligned} \tag{43}$$

where the sums extend over the central point and its nearest neighbours (N is the number of nearest neighbours plus one). This expression differs from the one proposed by LH (Eq. 7) in some 'weighting' factors and signs.

6. Conclusion

We have shown that the evolution rules of the CA model of LH can be interpreted in continuous time and space as a PDE of diffusive type — acting in a localized region, and subdued to an instability criterion —, together with a source term. Since the diffusive part of this equation has a trivial solution after the diffusive time has elapsed, the CA rules of LH implement directly this solution, namely flattened magnetic field in a bounded region.

The main assumptions on solar flares we have in this way revealed for the LH model are: (A) Every instability which possibly occurs has the same properties, namely the same characteristic length and the same diffusion time. (B) The diffusivity is an average quantity, without any spatial variations inside the unstable volume, modeling thus the complex dynamics of magnetic reconnection and diffusion in a simplified way. (C) The convective term is replaced by a simple random function, so the dynamics of the turbulent active region (plasma motion) is modeled in a strongly simplified way.

Despite of these limitations, the benefit of running a CA model for solar flares is that in this way an understanding of the global statistical properties of flares can be achieved. For instance, it provides an interpretation of the observed fragmented nature of the energy release process (see e.g. Vilmer and Trottet 1997, and references therein) as an avalanche-like phenomenon (chain-reactions of localized bursts). Such insights are far out of reach for a pure MHD-treatment of the problem: MHD and CA cannot replace each other, but they must be combined.

Our considerations allow extraction of more information from the so far existing CA models for solar flares, and to interpret some aspects in a more physical way:

1. Units: Since in Sect. 4 we have established the connection between the CA of Lu and Hamilton and a PDE, and in Sect. 5 the connection of this PDE to the induction equation has been shown, we can associate real physical units with the length scale l_0 (the grid size in the CA) and the diffusive time τ_d (the iteration time step in the CA). From this the units of the diffusivity, the magnetic field, the threshold, and the released energy can be derived. To introduce physical units in solar flare CA was not feasible, so far.
2. Diffusivity: Having real units (point 1), the values given in the literature for the diffusivity η can be used and discussed, e.g. the question can be addressed whether the values needed in the CA to be in a self-organized critical (SOC) state are physically reasonable or not, compared to anomalous diffusivity.
3. Released energy: Lu and Hamilton assume the amount of released energy to be $1/42\, l_0^4 \left|\nabla^2 \boldsymbol{B}\right|^2$ (Eqs. 7, 10, and 37). In the frame of MHD, an expression for the released energy should be in terms of the currents, or else it can be estimated by the difference in $\boldsymbol{B}^2/2\mu$ before and after a burst (flare), as given in Eq. (43).
4. Energy balance: If the previous point is clarified, then the emitted energy can be calculated (in physical units), and it can be compared to the injected energy, analyzing thus the energy balance of CA models.

Furthermore, on the basis of the given discussion, concrete suggestions of how CA-rules can be modified to include more physical insight (MHD) can be made, for instance:

1. The driver: Lu and Hamilton had replaced the convective term $\nabla \wedge (\boldsymbol{v} \wedge \boldsymbol{B})$ in the induction equation (Eq. 39) by a simple random function ($\boldsymbol{S}$ in Eq. 33). To use the convective term would mean to do full MHD, since also the momentum equation would have to be included, and so a treating of the whole problem with a CA would become as complex as to integrate the full PDEs (as e.g. Einaudi et al. 1996 have done). The question is whether this random loading term $\boldsymbol{S}$ of LH could be replaced by a description of the convective motion which is still simplified, but which catches more of the physical picture we have on the convective motion in the corona, in such a way, however, that it still is possible to reduce the problem to a CA. A possible set-up would be that, instead of random loading everywhere, the random loading is only from below (the photosphere), and thereafter the magnetic field is shuffled from site to site, e.g. through a term $\nabla \wedge (\boldsymbol{V} \wedge \boldsymbol{B})$, with $\boldsymbol{V}$ a random variable, varying from site to site and with a distribution function taken from Kolmogoroff's theory of turbulence.
2. Instability criterion: LH use $\left|\nabla^2 \boldsymbol{B}\right|$ as a critical quantity for the onset of an instability. A different approach would be to consider the slope of the magnetic field in some direction $\boldsymbol{n}$: $\left|(\boldsymbol{n}\nabla)\boldsymbol{B}\right|$, or the current $\boldsymbol{J} = \nabla \wedge \boldsymbol{B}/\mu$. Related to this question is a discussion of the magnitude and nature of the threshold (in physical units).
3. Released energy: An improved way of formulating the amount of released energy has been given as point 3, above.

We believe that the connection of MHD with CA models, for which we have given here a first example, will help to improve the global modeling of active regions, since insights accumulated in MHD may give guidelines to improve CA rules. Moreover, it is a step towards the combination of MHD and CA models, i.e. towards a model which incorporates the micro-physical as well as the global aspects of flares.

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