The Trivariate Weibull Distribution with Arbitrary Correlation

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Abstract— **In this paper an anlytical framework for analyzing the arbitrarily correlated trivariate Weibull distribution is introduced. For this distribution infinite series representations for the joint probability density function, the cumulative distribution function (CDF) and the moments are derived. Two special correlation cases of the distribution are studied: the exponential and the constant. These series representations are readily applicable to the performance analysis of a 3-branch selection combining (SC) receiver operating in a Weibull correlation fading environment and the outage probability is derived. The proposed mathematical analysis is complemented by various numerical results, showing the effects of fading severity, correlation and the power decay factor.**

I. Introduction

During the last years there has been a continuing interest in multivariate statistics for modelling and analyzing realistic wireless channels with correlated fading [1]. In diversity reception which as well-known reduces the effect of fading on the system's performance, is frequently assumed that the antennas are sufficiently separated so that the combined signals are independent of each other. However, this assumption is not always valid, e.g. for applications employing wireless terminals with insufficient antenna spacing for space and polarization diversity (mobile terminal, indoor base-station, *etc.*). It is also well-known that the correlation among the receiving channels results in a degradation of the diversity gain [2].

There are several fading correlation models, which correspond to certain correlation environments and communication scenaria. The most widely used are the constant and the exponential correlation models [1]. For the first one the correlation depends on the distance among the combining antennas and as a consequence it is more suitable for equidistant antennas. The second one, corresponds to the scenario of multichannel reception from equispaced diversity antennas. This model has been widely used for performance analysis of space diversity techniques [3], [4], [5] or multiple-input multiple-output (MIMO) systems [6]. The arbitrary correlation model [7], used in this paper, is more generic as it includes the two previously mentioned models as special cases.

Past works concerning multivariate distributions with arbitrary correlation can be found in [2], [7]–[9]. In [7] new infinite series representations for the joint probability density function (PDF) and the joint cumulative distribution function (CDF) of three and four arbitrarily correlated Rayleigh random variables were presented. In [2] useful closed-form expressions for the joint Nakagami-m multivariate PDF and CDF with arbitrary correlation were derived and the correlation matrix was approximated by a Green's matrix. Similarly in [8], the Green's matrix was used to approximate the correlation matrix of L branch selection combining (SC) receivers with arbitrary correlation and the outage probability for lognormal fading channels has been obtained. In [9] expressions for multivariate Rayleigh and exponential PDFs generated from correlated Gaussian random variables were presented, while a general expression for the multivariate exponential characteristic function (CF), in terms of determinants, was also derived.

The Weibull distribution, although it was originally used in reliability and failure data analysis, it has been recently considered for wireless digital communication systems. The main motivation for this is the fact that it exhibits a very good fit to experimental fading channel measurements for both indoor and outdoor terrestrial radio propagation environments [10], [11]. Additionally, in [12], it was argued that the Weibull distribution could also been considered as a more generic channel model for land-mobile satellite systems. In a recent paper [13], the joint PDF, CDF and the moment-generating function (MGF) for the bivariate Weibull distribution have been analytically presented. The multivariate Weibull distribution has also been studied for the exponential and the constant correlation case with equal average fading powers. However, to the best of the authors' knowledge, an analytical framework for the trivariate Weibull distribution with arbitrary correlation, has not yet been presented in the open technical literature. Thus, this is the subject of our paper, whereby we apply the developed analytical framework to the performance evaluation of a 3-branch SC receiver.

II. Statistical Characteristics

Let h_ℓ , $\ell = 1, 2, 3$ be the complex envelope of the Weibull fading model, written as

$$h_\ell = (X_\ell + jY_\ell)^{2/\beta_\ell} = G_\ell^{2/\beta_\ell} \tag{1}$$

where X_ℓ and Y_ℓ are the Gaussian in-phase and quadrature elements of the multipath components [13] and j is the imaginary operator. By representing Z_ℓ as the magnitude of h_ℓ, i.e., $Z_\ell = |h_\ell|$, where $|\cdot|$ denotes absolute value, Z_ℓ can be expressed as a power transformation of a Rayleigh distributed random variable (RV) $R_\ell = |X_\ell + jY_\ell|$ as [14]

$$Z_\ell = R_\ell^{2/\beta_\ell}. \tag{2}$$

Let $G = \{G_1, G_2, G_3\}$ be joint complex Gaussian RVs with zero means and positive definite covariance matrix $\mathbf{\Psi}$, with elements $\psi_{i\kappa} = E\langle G_i G_\kappa^*\rangle$, where $E\langle\cdot\rangle$ denotes expectation. By applying (2) in the infinite series representation of the Rayleigh trivariate distribution [7, eq. (5)], the trivariate Weibull distribution can be expressed as

$$\begin{aligned} f_{Z_1,Z_2,Z_3}(z_1,z_2,z_3) &= \frac{\beta_1\beta_2\beta_3 \det(\mathbf{\Phi})}{z_1^{(2-\beta_1)/2} z_2^{(2-\beta_2)/2} z_3^{(2-\beta_3)/2}} \\ &\times \exp\left[-\left(z_1^{\beta_1}\phi_{11} + z_2^{\beta_2}\phi_{22} + z_3^{\beta_3}\phi_{33}\right)\right] \\ &\times \sum_{k=0}^{\infty} \epsilon_k (-1)^k \cos(k\chi) \\ &\times \sum_{\ell,m,n=0}^{\infty} \frac{|\phi_{12}|^{2\ell+k}}{\ell!(\ell+k)!} \frac{|\phi_{23}|^{2m+k}}{m!(m+k)!} \frac{|\phi_{31}|^{2n+k}}{n!(n+k)!} \\ &\times z_1^{\beta_1(\ell+n+k)+\beta_1/2} z_2^{\beta_2(\ell+m+k)+\beta_2/2} z_3^{\beta_3(m+n+k)+\beta_3/2} \end{aligned} \tag{3}$$

where ϵ_k is the Neumann factor ($\epsilon_0 = 1, \epsilon_k = 2$ for $k = 1, 2, \cdots$), $\chi = \chi_{12} + \chi_{23} + \chi_{31}$ and $\mathbf{\Phi}$ is the inverse covariance matrix in the case of the trivariate Weibull distribution, given by

$$\mathbf{\Phi} = \mathbf{\Psi}^{-1} = \begin{bmatrix} \phi_{11}, & \phi_{12}, & \phi_{13} \\ \phi_{12}^*, & \phi_{22}, & \phi_{23} \\ \phi_{13}^*, & \phi_{23}^*, & \phi_{33} \end{bmatrix} \tag{4}$$

for $\phi_{i\kappa} = |\phi_{i\kappa}| \exp(j\chi_{i\kappa})$ and $i, \kappa \in \{1, 2, 3\}$.

By integrating (3) an infinite series representation for the CDF of Z_1, Z_2, Z_3 is derived as

$$\begin{aligned} F_{Z_1,Z_2,Z_3}(z_1,z_2,z_3) &= \frac{\det(\mathbf{\Phi})}{\phi_{11}\phi_{22}\phi_{33}} \sum_{k=0}^{\infty} \epsilon_k (-1)^k \cos(k\chi) \\ &\times \sum_{\ell,m,n=0}^{\infty} C_{\ell,m,k}\, \nu_{12}^{\ell+k/2} \nu_{23}^{m+k/2} \nu_{31}^{n+k/2} \\ &\times \gamma\left(\delta_1, z_1^{\beta_1}\phi_{11}\right) \gamma\left(\delta_2, z_2^{\beta_2}\phi_{22}\right) \gamma\left(\delta_3, z_3^{\beta_3}\phi_{33}\right) \end{aligned} \tag{5}$$

where

$$C_{\ell,m,k} = \frac{1}{\ell!(\ell+k)!m!(m+k)!n!(n+k)!},$$

$$\nu_{i\kappa} = \frac{|\phi_{i\kappa}|^2}{\phi_{ii}\phi_{\kappa\kappa}},$$

and $\delta_1 = \ell + n + k + 1$, $\delta_2 = m + \ell + k + 1$, $\delta_3 = n + m + k + 1$ with $\gamma(\cdot,\cdot)$ denoting the incomplete lower Gamma function [15, eq. (3.381/1)].

Moments are a useful statistical tool to characterize a distribution. Using (3), the product moments of the $(\alpha + \eta + \vartheta)th$ order of Z_1, Z_2 and Z_3 can been derived as

$$\begin{aligned} E\langle Z_1^\alpha, Z_2^\eta, Z_3^\vartheta\rangle &= \frac{\det(\mathbf{\Phi})}{\phi_{11}^{1+\alpha/\beta_1} \phi_{22}^{1+\eta/\beta_2} \phi_{33}^{1+\vartheta/\beta_3}} \\ &\times \sum_{k=0}^{\infty} \epsilon_k (-1)^k \cos(k\chi) \sum_{\ell,m,n=0}^{\infty} C_{\ell,m,k}\, \nu_{12}^{\ell+k/2} \nu_{23}^{m+k/2} \nu_{31}^{n+k/2} \\ &\times \Gamma\left(\delta_1 + \frac{\alpha}{\beta_1}\right) \Gamma\left(\delta_2 + \frac{\eta}{\beta_2}\right) \Gamma\left(\delta_3 + \frac{\vartheta}{\beta_3}\right) \end{aligned} \tag{6}$$

with $\Gamma(\cdot)$ denoting the gamma function [15, eq. (8.310/1)].

Using the theoretical analysis above, two previously used spatial correlation models can be studied as special cases.

A. Constant Correlation Model

The constant correlation model was first proposed by Aalo in [16] and is valid for a set of closely placed antennas. Its normalized correlation matrix consists of the elements $\psi_{i\kappa} = \rho(i \neq \kappa)$ and $\psi_{ii} = 1$, where $-1/2 \leq \rho < 1$. Moreover, it can been also shown that in this case $\chi = \chi_{12} + \chi_{23} + \chi_{31} = 3\pi$ [7]. As a consequence, (5) for the constant correlation case reduces to

$$F_{Z_1,Z_2,Z_3}(z_1, z_2, z_3) = \frac{(1-\rho)(1+2\rho)^2}{(1+\rho)^3} \times \sum_{k=0}^{\infty} \epsilon_k \sum_{\ell,m,n=0}^{\infty} C_{\ell,m,k} \left(\frac{\rho}{1+\rho}\right)^{\delta_1+\delta_2+\delta_3-3} \lambda_1\lambda_2\lambda_3 \tag{7}$$

where $\lambda_\ell = \gamma\left(\delta_\ell, \frac{(1+\rho)z_\ell^{\beta_\ell}}{(1+\rho-2\rho^2)\Omega_\ell}\right)$.

Note here that for $\Omega_1 = \Omega_2 = \Omega_3 = \Omega$, (7) agrees with the analysis presented in [13] for the multivariate Weibull distribution with constant correlation and for identical average powers.

B. Exponential Correlation Model

The exponential correlation model is valid for a set of equally spaced linear antenna arrays, in which the correlation among the pairs of combined signals decreases as the spacing between the antenna increases [1]. The normalized correlation matrix of this model is given by $\psi_{i\kappa} = \rho^{|i-\kappa|}$, where $0 \leq \rho < 1$. Moreover, $\phi_{31} = \phi_{13} = 0$ [7]. Thus the joint CDF of (5) simplifies to

$$\begin{aligned} F_{Z_1,Z_2,Z_3}(z_1, z_2, z_3) &= \frac{(1-\rho^2)}{(1+\rho^2)} \\ &\times \sum_{\ell,m=0}^{\infty} \frac{1}{(\ell!)^2(m!)^2}\left(\frac{\rho^2}{1+\rho^2}\right)^{\ell+m} \\ &\times \gamma\left(\ell+1, \frac{z_1^{\beta_1}}{(1-\rho^2)\Omega_1}\right)\gamma\left(\ell+m+1, \frac{(1+\rho^2)z_2^{\beta_2}}{(1-\rho^2)\Omega_2}\right) \\ &\times \gamma\left(m+1, \frac{z_3^{\beta_3}}{(1-\rho^2)\Omega_3}\right). \end{aligned} \tag{8}$$

Note again that for the simple case where $\Omega_1 = \Omega_2 = \Omega_3 = \Omega$, (8) leads to the same results as [13, eq. (28)] for $L = 3$.

III. Outage Probability of SC Receivers

Let us now consider a SC diversity receiver with three branches operating over correlated Weibull fading channels. Let $\zeta_\ell = wh_\ell + n_\ell$ be the received in the ℓth branch baseband signal, where w is the complex transmitted symbol, with $E_s = E\langle|w|^2\rangle$ being the transmitted average symbol's energy and n_ℓ is the additive white Gaussian noise (AWGN) with single-sided power spectral density N_0.

The instantaneous SNR per symbol of the ℓth diversity channel can be written as

$$\gamma_\ell = Z_\ell^2 \frac{E_s}{N_0}. \tag{9}$$

Moreover, the corresponding average SNR is expressed as

$$\overline{\gamma}_\ell = E\langle Z_\ell^2\rangle \frac{E_s}{N_0} = \Gamma(d_{2,\ell})\Omega_\ell^{2/\beta_\ell}\frac{E_s}{N_0} \tag{10}$$

where $d_{\tau,\ell} = 1+\tau/\beta_\ell$ with τ taking non-negative values.

Expressions for the statistics of γ_ℓ can be easily derived by replacing β_ℓ with $\beta_\ell/2$ and Ω_ℓ with $(\alpha_\ell\overline{\gamma}_\ell)^{\beta_\ell/2}$, in the corresponding expressions for the fading envelope Z_ℓ. Thus using (7), the CDF of the SNR for the constant correlation case can be expressed as

$$F_Z(\gamma_1, \gamma_2, \gamma_3) = \frac{(1-\rho)(1+2\rho)^2}{(1+\rho)^3} \times \sum_{k=0}^{\infty} \epsilon_k \sum_{\ell,m,n=0}^{\infty} C_{\ell,m,k} \left(\frac{\rho}{1+\rho}\right)^{\delta_1+\delta_2+\delta_3-3} \mu_1\mu_3\mu_3 \tag{11}$$

where $\mu_\ell = \gamma\left(\delta_\ell, \frac{(1+\rho)\gamma_\ell^{\beta_\ell/2}}{(1+\rho-2\rho^2)(\Xi_\ell\overline{\gamma}_\ell)^{\beta_\ell/2}}\right)$.

Furthermore, using (8) the corresponding CDF for the exponential correlation case can be derived as

$$\begin{aligned} F_Z(\gamma_1, \gamma_2, \gamma_3) &= \frac{(1-\rho^2)}{(1+\rho^2)} \\ &\times \sum_{\ell,m=0}^{\infty} \frac{1}{(\ell!)^2(m!)^2}\left(\frac{\rho^2}{1+\rho^2}\right)^{\ell+m} \\ &\times \gamma\left(\ell+1, \frac{\gamma_1^{\beta_1/2}}{(1-\rho^2)(\Xi_1\overline{\gamma}_1)^{\beta_1/2}}\right) \\ &\times \gamma\left(\ell+m+1, \frac{(1+\rho^2)\gamma_2^{\beta_2/2}}{(1-\rho^2)(\Xi_2\overline{\gamma}_2)^{\beta_2/2}}\right) \\ &\times \gamma\left(m+1, \frac{\gamma_3^{\beta_3/2}}{(1-\rho^2)(\Xi_3\overline{\gamma}_3)^{\beta_3/2}}\right). \end{aligned} \tag{12}$$

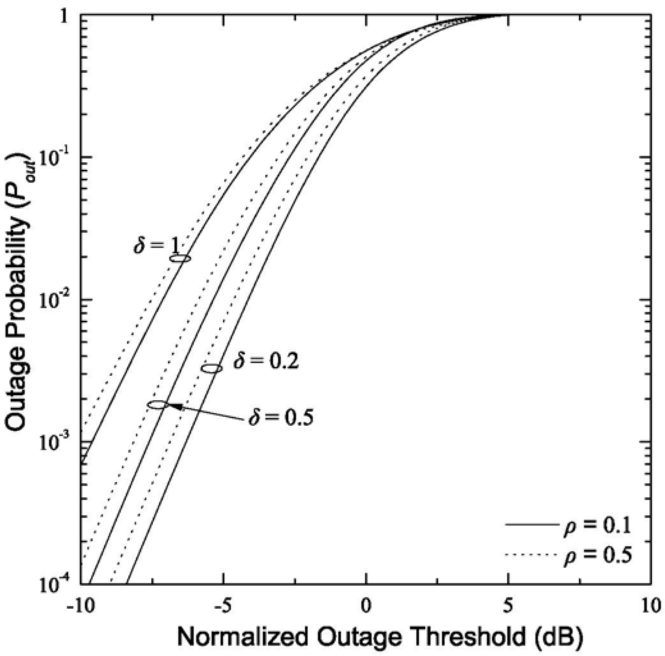

Fig. 1. Outage probability versus the normalized average input SNR for several values of δ and ρ.

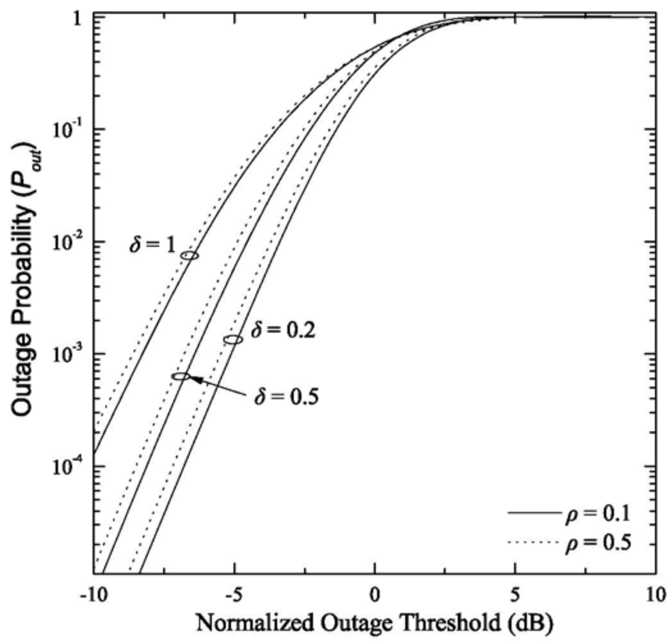

Fig. 2. Outage probability versus the normalized average input SNR for several values of δ and ρ.

For the SC receiver, the instantaneous per symbol SNR at the output of a triple-branch receiver of this type, will be the one with the highest instantaneous value between the three branches [17]

$$\gamma_{sc} = \max\{\gamma_1, \gamma_2, \gamma_3\}. \tag{13}$$

P_{out} is defined as the probability that the instantaneous error probability exceeds a specified value or equivalently the probability that the receiver output SNR, γ_{sc}, falls below a certain specified threshold, γ_{th} [1]. P_{out} can be derived by replacing γ_1, γ_2 and γ_3 with γ_{th} in (11) and (12), i.e.,

$$P_{out}(\gamma_{th}) = F_Z(\gamma_{th}). \tag{14}$$

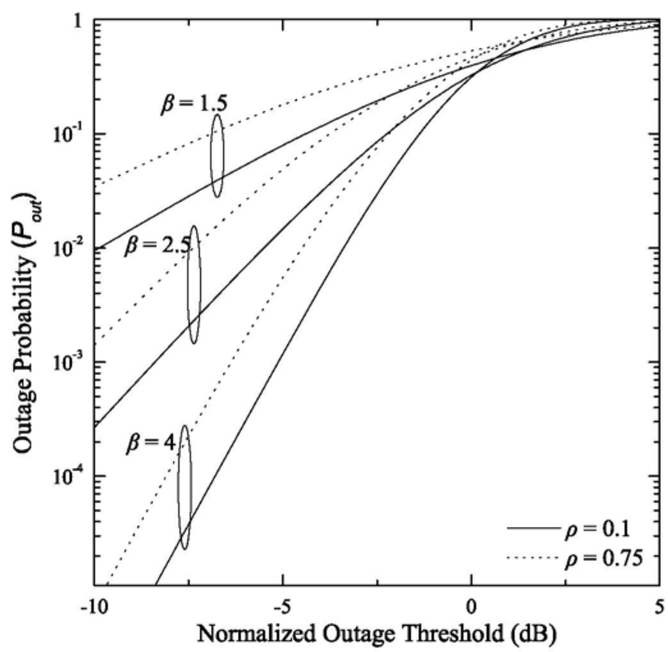

Fig. 3. Outage probability versus the normalized average input SNR for several values of β and ρ.

IV. Performance Evaluation Results

In this section we use the previous mathematical analysis to present various performance evaluation results for the P_{out} of SC diversity receivers operating over correlated Weibull fading channels. For the convenience of the presentation, but without any loss of generality, it is assumed that $\beta_\ell = \beta \, \forall \, \ell$. Furthermore, we assume non-identical distributed Weibull channels, i.e., $\overline{\gamma}_\ell = \overline{\gamma}_1 \exp(-(\ell-1)\delta)$ where δ is the power decay factor. Our performance evaluation results have been obtained by numerically evaluating (14).

In Fig. 1, P_{out} is plotted for the case of the constant correlation model, as a function of the first branch normalized outage threshold $\gamma_{th}/\overline{\gamma}_1$ for a triple-branch SC, assuming $\beta = 3.3$ and for different values of δ and the correlation coefficient ρ. In Fig. 2, P_{out} is plotted for the exponential case and for the same parameters as in

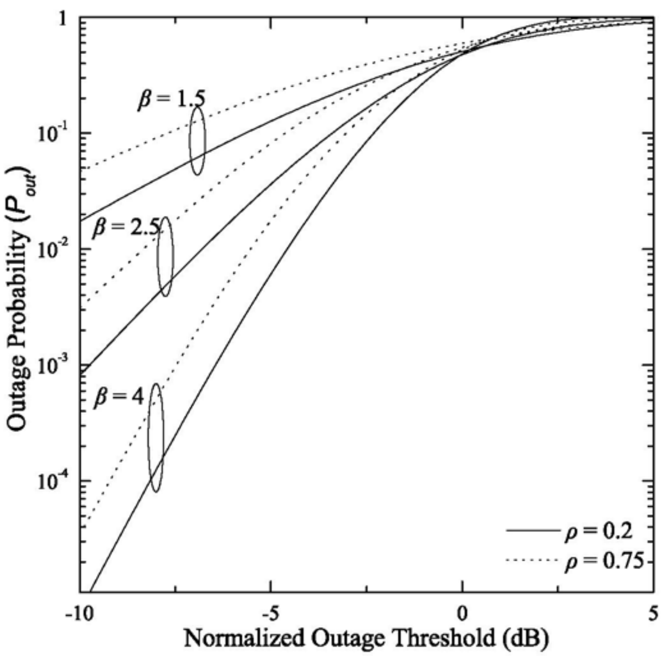

Fig. 4. Outage probability versus the normalized average input SNR for several values of β and ρ.

Fig. 1 but considering $\beta = 4$.

The obtained results in both cases show clearly that P_{out} degrades with an increase of ρ, $\gamma_{th}/\overline{\gamma}_1$ and δ.

In Fig. 3, P_{out} is also plotted for the case of the constant correlation model, as a function of the first branch normalized threshold, assuming $\delta = 0.2$, for different values of ρ and β. Again an increase of ρ, $\gamma_{th}/\overline{\gamma}_1$ and/or the fading severity, leads to the degradation of P_{out}.

Finally in Fig. 4, assuming that $\delta = 0.5$ and for the same parameters as in Fig. 3, the same findings are also valid for the case of exponential correlation.

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