Information Outage Probability of Orthogonal Space-Time Block Codes over Hoyt Distributed Fading Channels

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***Abstract*— In this paper the information outage probability (IOP) of orthogonal space-time block coded multiple-input multiple-output (MIMO) systems operating over Hoyt distributed fading channels is investigated. In our analysis, we consider two separate fading cases. For independent identically distributed (i.i.d.) fading channels, exact closed-form expressions of the IOP are derived. For non-identically Hoyt distributed fading, we express the problem in a different equivalent form, and propose the use of the saddlepoint method to approximate the IOP. As verified by extensive computer simulations, the saddlepoint approximation method provides very accurate results, making it an approach that could potentially be applied in performance evaluation of wireless communication systems.**

I. Introduction

Due to its many attractive features, diversity transmission using orthogonal space-time block codes (OSTBC) has gained considerable attention in recent years [1]. Among others, this approach offers full spatial diversity and maximum likelihood performance with linear decoding complexity. Performance evaluation of OSTBC MIMO systems can be carried out by assuming an appropriate statistical fading model for the propagation environment. In the past, various performance analysis results have been presented for OSTBC MIMO operating over Rayleigh, Nakagami-m [2], and Nakagami-n (Rice) [3] fading channels. However, only a few relevant results exist for the Nakagami-q (Hoyt) distribution. As shown in many studies, the Hoyt fading model provides a very accurate fit to experimental channel measurements in various telecommunications applications. For instance, in [4] this model has been used in outage analysis of cellular mobile radio systems. Similarly, the Hoyt distribution can be considered an accurate fading model for satellite links with strong ionospheric scintillation, ranging from one-sided Gaussian fading to Rayleigh fading [5].

In this work, the information outage probability (IOP) of OSTBC MIMO systems operating over independent but not necessarily identical Hoyt distributed fading channels is investigated. As will be shown in the following analysis, the IOP can be computed by determining the distribution of the sum of squares of a number of Hoyt distributed random variables (RVs). Such distribution is described in [6] by an infinite series, while in [7] an exact closed-form expression is derived, when the number of i.i.d. Hoyt RVs is even. In this paper, the IOP is determined irrespective of the number of transmit and receive antennas of the MIMO system, i.e., the number of Hoyt RVs. For i.i.d. Hoyt fading exact closed-form expressions are derived, avoiding the use of infinite series. For independent non-identically distributed Hoyt fading, the so-called saddlepoint method [8] is applied to approximate the required pdf. Although the saddlepoint approximation method has been extensively used in statistics, its use in performance evaluation of wireless communication systems has been rather limited. Extensive computer simulations verify our theoretical analysis with respect to both exact closed-form and approximation results.

The outline of the paper is as follows. In Section II the problem is formulated and in Section III the IOP over OSTBC MIMO Hoyt fading channels is derived. Simulation results are presented in Section IV, while some concluding remarks are given in Section V.

II. Problem Statement

Let us consider an OSTBC MIMO system with n_t transmit and n_r receive antennas. It can then be shown that the normalized capacity of the system, measured in bits/second/Hertz, has the form [1]

$$C = R\ \log_2\left(1 + \frac{\|H\|_F^2}{n_t R}\frac{S}{N_0}\right) \tag{1}$$

where R is the rate of the OSTBC, S is the average symbol power allocated to the transmit antennas and N is the noise power that is common for all receive antennas. In addition,

$$\|H\|_F^2 = \sum_{i=1}^{n_t}\sum_{j=1}^{n_r} |h_{i,j}|^2 \tag{2}$$

is the square of the Frobenius norm of the channel matrix, where $|h_{i,j}|$ stands for the amplitude of the fading coefficient between the ith transmit and the jth receive antenna. It is assumed that $|h_{i,j}|$ are independent Hoyt distributed RVs,

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whose pdf is given by [9]

$$p_{|h_{i,j}|}(r) = \frac{\left(1+q_{i,j}^2\right) r}{q_{i,j}\Omega_{i,j}} e^{-\frac{\left(1+q_{i,j}^2\right)^2 r^2}{4q_{i,j}^2\Omega_{i,j}}} I_0\left(\frac{\left(1-q_{i,j}^4\right) r^2}{4q_{i,j}^2\Omega_{i,j}}\right) \tag{3}$$

for $r \geq 0$, where $\Omega_{i,j} = E[|h_{i,j}|^2]$, $I_0(\cdot)$ is the zeroth order modified Bessel function of the first kind and parameter $q_{i,j}$ may take different values among the channel fading coefficients. It is then easy to derive that the RV $z_{i,j} = |h_{i,j}|^2$ will be distributed according to [9]

$$p_{z_{i,j}}(r) = \frac{\left(1+q_{ij}^2\right)}{2q_{ij}\Omega_{ij}} e^{-\frac{\left(1+q_{i,j}^2\right)^2 r}{4q_{i,j}^2\Omega_{i,j}}} I_0\left(\frac{\left(1-q_{ij}^4\right) r}{4q_{ij}^2\Omega_{ij}}\right). \tag{4}$$

Moreover, the moment generating function (MGF) of $z_{i,j}$ will be given by the following expression [9]

$$M_{z_{i,j}}(s) = \frac{1+q_{i,j}^2}{2\Omega_{i,j}q_{i,j}} \frac{1}{\sqrt{\left(s-\frac{1+q_{i,j}^2}{2\Omega_{i,j}q_{i,j}^2}\right)\left(s-\frac{1+q_{i,j}^2}{2\Omega_{i,j}}\right)}}. \tag{5}$$

An important performance evaluation criterion of wireless communications systems is the so-called information outage probability [10], defined as the probability that a transmission rate C_0 cannot be supported, i.e.,

$$P_{out}(C_0) = \Pr(C < C_0) = \Pr\left(\|H\|_F^2 < A\right). \tag{6}$$

From (1) and (6), A is readily expressed as

$$A = \frac{Nn_tR}{S}\left(2^{\frac{C_0}{R}} - 1\right). \tag{7}$$

We deduce from (6) that in order to calculate IOP, the cumulative distribution function (CDF) of the squared Frobenius norm of the channel matrix is required. More specifically, from (2), the CDF of the sum of n_tn_r squared independent Hoyt RVs must be determined.

III. IOP of OSTBC over Hoyt distributed channels

In this section, exact closed-form expressions of the IOP for i.i.d. Hoyt distributed fading in OSTBC MIMO systems are derived. In the sequel, approximations of the IOP for non-identically Hoyt distributed fading are also provided.

A. i.i.d. Hoyt fading

From (5), assuming that $\Omega_{i,j} = \Omega$ and $q_{i,j} = q$ $\forall$ i and j, the MGF of the sum ς of n squared Hoyt RVs can be written as follows [1]

$$M_\varsigma(s) = \left(\frac{1+q^2}{2\Omega q}\right)^n \frac{1}{\left(s-\frac{1+q^2}{2\Omega q^2}\right)^{\frac{n}{2}}\left(s-\frac{1+q^2}{2\Omega}\right)^{\frac{n}{2}}}. \tag{8}$$

By properly exploiting the relation of the MGF with the Laplace transform, it can be shown that the pdf of the sum of i.i.d. squared Hoyt RVs has the form [11]

$$p_\varsigma(r) = ar^{\frac{n-1}{2}}e^{-br}I_{\frac{n-1}{2}}(cr) \tag{9}$$

with

$$a = \left(\frac{1+q^2}{2\Omega q}\right)^{\frac{n+1}{2}} \frac{\sqrt{\pi}}{\Gamma\left(\frac{n}{2}\right)}\left(\frac{q}{1-q^2}\right)^{\frac{n-1}{2}} \tag{10}$$

$$b = \frac{\left(1+q^2\right)^2}{4\Omega q^2} \tag{11}$$

and

$$c = \frac{1-q^4}{4\Omega q^2}. \tag{12}$$

In (9), $I_p(\cdot)$ is the p-th order modified Bessel function of the first kind. The pdf of (9) is of the same form of the McKay distribution [12]. It also appears in [11], [6] as the pdf of a squared η-μ or λ-μ distributed RV respectively. The CDF of $p_\varsigma(r)$ is given by

$$F_\varsigma(x) = \int_0^x ar^{\frac{n-1}{2}}e^{-br}I_{\frac{n-1}{2}}(cr)\,dr. \tag{13}$$

In [6] this CDF is expressed in a form comprising infinite series. Nevertheless, closed-form expressions for the CDF can be obtained if the fact that n is an integer is taken into consideration. More specifically, we distinguish the following two cases.

1) CDF for n even: In this case, by using the properties of the modified Bessel function of the first kind of order $(n-1)/2$ with n even, [13], the CDF in (13) can be expressed as:

$$\begin{aligned} F_\varsigma(x) =& \frac{a}{\sqrt{2\pi c}}\sum_{k=0}^{\frac{n}{2}-1} u_k \int_0^x r^{\frac{n}{2}-k-1}e^{-(b-c)r}dr \\ &+\frac{a}{\sqrt{2\pi c}}\sum_{k=0}^{\frac{n}{2}-1} v_k \int_0^x r^{\frac{n}{2}-k-1}e^{-(b+c)r}dr \end{aligned} \tag{14}$$

where

$$u_k = \frac{(-1)^k\left(\frac{n}{2}-1+k\right)!}{k!\left(\frac{n}{2}-1-k\right)!\,(2c)^k} \tag{15}$$

and

$$v_k = \frac{(-1)^{\frac{n}{2}}\left(\frac{n}{2}-1+k\right)!}{k!\left(\frac{n}{2}-1-k\right)!\,(2c)^k}. \tag{16}$$

After some algebraic manipulations $F_\varsigma(x)$ can be presented in a more compact form as

$$\begin{aligned} F_\varsigma(x) =& \sum_{k=0}^{\frac{n}{2}-1} d_k\gamma\left(\frac{n}{2}-k, \frac{1+q^2}{2\Omega}x\right) \\ &+\sum_{k=0}^{\frac{n}{2}-1} g_k\gamma\left(\frac{n}{2}-k, \frac{1+q^2}{2\Omega q^2}x\right) \end{aligned} \tag{17}$$

where $\gamma(\cdot)$ is the incomplete gamma function [14], while

$$d_k = (-1)^k \frac{\left(\frac{n}{2}-1+k\right)!q^{2k}}{(k)!\left(\frac{n}{2}-1-k\right)!\Gamma\left(\frac{n}{2}\right)(1-q^2)^{\frac{n}{2}+k}} \tag{18}$$

and

$$g_k = (-1)^{\frac{n}{2}} \frac{\left(\frac{n}{2}-1+k\right)!q^{n}}{(k)!\left(\frac{n}{2}-1-k\right)!\Gamma\left(\frac{n}{2}\right)(1-q^2)^{\frac{n}{2}+k}}. \tag{19}$$

[1] In our case $n = n_tn_r$.

Note that an equivalent expression of the CDF for n even was presented in [7] following though a different approach.

As a result, the IOP for MIMO OSTBC for $n = n_t n_r$ even can be expressed as

$$P_{out}(C_0) = \sum_{k=0}^{\frac{n}{2}-1} d_k \gamma\left(\frac{n}{2} - k, \frac{1+q^2}{2\Omega}A\right) + \sum_{k=0}^{\frac{n}{2}-1} g_k \gamma\left(\frac{n}{2} - k, \frac{1+q^2}{2\Omega q^2}A\right) \tag{20}$$

with A given by (7) and d_k, g_k by (18), (19) respectively.

2) CDF for n odd: In case of n odd, although no expansion of the modified Bessel function as a finite sum seems to be possible, we can still derive the form of the CDF of the sum of i.i.d. squared Hoyt RVs by exploiting integrals of the form

$$Z_m(u, x) = \int_0^x z^m e^{-uz} I_m(z)\, dz. \tag{21}$$

Specifically, combining equations [14, eq. (11.3.4)], [14, eq. (11.3.5)], $Z_m(u, x)$ can be written as

$$Z_m(u, x) = -\frac{u}{u^2-1} e^{-ux} x^m I_m(x) - \frac{1}{u^2-1} e^{-ux} x^m I_{m-1}(x) + \frac{2m-1}{u^2-1} Z_{m-1}(u, x). \tag{22}$$

Hence, for m taking positive integer values, recursive application of expression (22) leads to

$$Z_m(u, x) = \sum_{k=1}^{m} \delta_k x^k e^{-ux} I_k(x) + \sum_{k=1}^{m} \zeta_k x^k e^{-ux} I_{k-1}(x) + \varepsilon Z_0(u, x) \tag{23}$$

where

$$\delta_k = -\frac{u}{(u^2-1)^{m-k+1}} \frac{\prod_{i=1}^{m}(2i-1)}{\prod_{j=1}^{k}(2j-1)} \tag{24}$$

$$\zeta_k = -\frac{1}{(u^2-1)^{m-k+1}} \frac{\prod_{i=1}^{m}(2i-1)}{\prod_{j=1}^{k}(2j-1)} \tag{25}$$

and

$$\varepsilon = \frac{\prod_{i=1}^{m}(2i-1)}{(u^2-1)^m}. \tag{26}$$

Additionally, it is rather straightforward to prove that $Z_0(x)$ is written as

$$Z_0(u, x) = \frac{1}{u} I_e\left(\frac{1}{u}, ux\right) \tag{27}$$

where $I_e(\cdot)$ is the Rice I_e function [15]. As a result, we have that

$$Z_m(u, x) = \sum_{k=1}^{m} \delta_k x^k e^{-ux} I_k(x) + \sum_{k=1}^{m} \zeta_k x^k e^{-ux} I_{k-1}(x) + \frac{\varepsilon}{u} I_e\left(\frac{1}{u}, ux\right). \tag{28}$$

From (6),(13) and (21) we have

$$P_{out}(C_0) = \frac{a}{c^{\frac{n+1}{2}}} Z_{\frac{n-1}{2}}\left(\frac{b}{c}, cA\right) \tag{29}$$

and thus the IOP for $n = n_t n_r$ odd is expressed in closed form according to (28). Note that in practice the value of the I_e function can be computed using its relation to the Marcum-Q function, which is included in most mathematical software packages, [16], i.e.,

$$2Q\left(\sqrt{U+W}, \sqrt{U-W}\right) = 1 + \frac{W}{U} I_e\left(\frac{V}{U}, U\right) + e^{-U} I_0(V) \tag{30}$$

where $W = \sqrt{U^2 - V^2}$ and $U \geq |V|$. Note that due to the relation $c \leq b$, (30) can be applied in our case.

B. Non-identically distributed Hoyt fading

It is known that a squared Hoyt RV can be expressed as the sum of squares of two independent zero-mean Gaussian RVs with different variances [17]. More specifically, the relation between the q and Ω parameters of a Hoyt RV and the variances σ_1 and σ_2 of the underlying Gaussian RVs is described by the equations

$$\sigma_1 = \sqrt{\frac{\Omega}{1+q^2}} \quad \text{and} \quad \sigma_2 = \sqrt{\frac{\Omega q^2}{1+q^2}}. \tag{31}$$

Therefore, for independent $|h_{i,j}|$, (2) can be written as

$$\|H\|_F^2 \cong \sum_{k=1}^{2n} \sigma_k^2 w_k^2 \tag{32}$$

where $n = n_t n_r$, w_k's are standard independent normal RVs, σ_k's are related to the parameters of the corresponding Hoyt RVs as described in (31) and '$\cong$' denotes equality in distribution. Hence, in order to compute the IOP, calculation of the CDF of a quadratic form in standard normal RVs is required. Such a CDF is known to have an exact form involving an infinite series expansion. Thus, for practical reasons a number of approximation methods have been developed, as described in [18]. In our analysis, the so-called saddlepoint approximation method, which provides quite accurate results with respect to the desired CDF, is used.

Although the saddlepoint approximation was introduced more than fifty years ago, it has only recently been proposed for the approximation of the distribution of quadratic forms in normal RVs [19]. A thorough description of the saddlepoint approximation method can be found in [8]. Using this approach, we can approximate the value of the CDF at any point either by using the so-called Lugannani-Rice formula [20] or the Barndorff-Nielsen formula [21]. Although both formulas provide quite similar results [19], the Barndorff-Nielsen formula was utilized. According to this approach, the CDF of a quadratic form in normal RVs at a point x is approximated by the quantity

$$F_\varsigma(x) = \Phi\left\{u + \frac{1}{u}\log\left(\frac{v}{u}\right)\right\} \tag{33}$$

where $\Phi(\cdot)$ is the CDF of a standard normal RV, and u, v are defined as

$$u = sign(z)\sqrt{2(zx - f(z))} \quad \text{and} \quad v = z\sqrt{f^{(2)}(z)} \tag{34}$$

In the last equation $f(\cdot)$, $f^{(i)}(\cdot)$ are the cumulant generating function of the quadratic form and its ith order derivative, respectively, and z is the saddlepoint, i.e., the root of the equation

$$f^{(1)}(z) = x. \tag{35}$$

In the case of a quadratic form in normal RVs such as the one given in (32), the cumulant generating function will be defined as

$$f(z) = -\frac{1}{2}\sum_{k=1}^{2n}\log\left(1 - 2z\sigma_k^2\right) \tag{36}$$

and the saddlepoint z will be given by the root of the following equation

$$\sum_{k=1}^{2n}\frac{\sigma_k^2}{1 - 2\sigma_k^2 z} = x. \tag{37}$$

It is clear that the saddlepoint equation (37) has more than one roots. Neverhteless, only one of them belongs to the domain of the cumulant generating function and this root will be the saddlepoint. Specifically, the saddlepoint will be the root z of (37) that satisfies $z < 1/(2\theta)$ where $\theta = \max\left\{\sigma_1^2, \sigma_2^2, \ldots, \sigma_{2n_tn_r}^2\right\}$ [19]. More details on the existence and uniqueness of the saddlepoint can be found in [8].

Having defined the saddlepoint, the IOP for any A can be approximated by setting $x = A$ in (37) in order to compute the saddlepoint, which is then substituted in (34) to obtain the u, v in (33).

IV. Performance Results

In this section theoretical results obtained using (29) and (33) along with computer simulations results are presented. In Fig. 1, IOP is plotted versus the transmission rate C_0 for a 3×1 and a 2×2 MIMO OSTBC system with i.i.d. and independent non-identical Hoyt fading respectively. For the case of a 3×1 MIMO system, a $3/4$ rate OSTBC is considered with the SNR set to 15dB. The i.i.d. channel parameters are $(\Omega, q) = (1, 0.20)$. For the 2×2 MIMO OSTBC system the Alamouti scheme is considered with SNR equal to 15dB. The parameters of the four non-identical channels are $(\Omega_{1,1}, q_{1,1}) = (1.22, 0.69)$, $(\Omega_{1,2}, q_{1,2}) = (0.79, 0.94)$, $(\Omega_{2,1}, q_{2,1}) = (1.21, 0.78)$ and $(\Omega_{2,2}q_{2,2}) = (0.78, 0.54)$. We observe that the corresponding theoretical and experimental

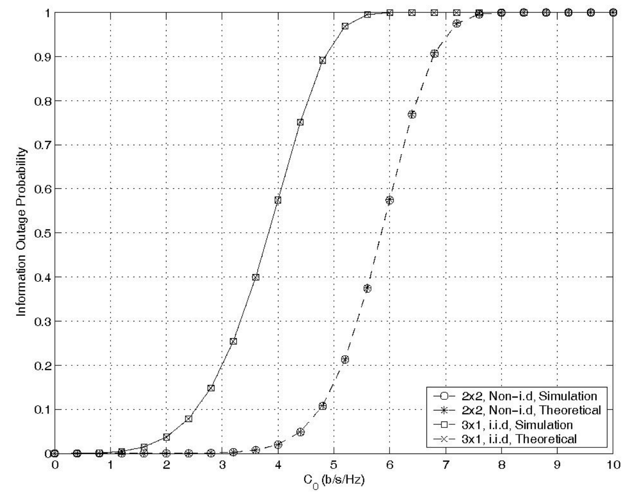

Fig. 1. IOP of OSTBC Hoyt fading

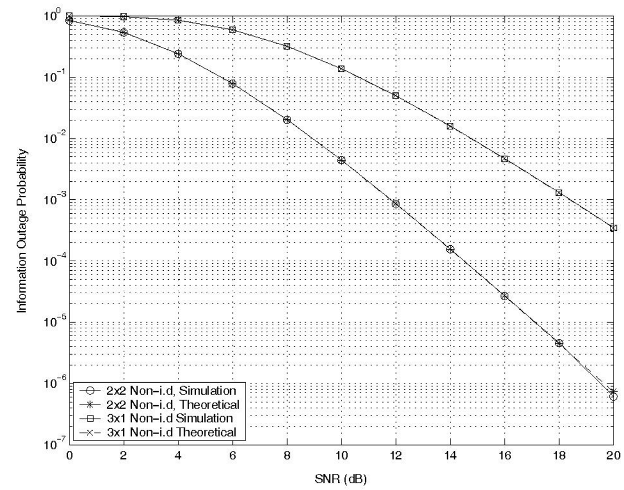

Fig. 2. IOP vs SNR for $C_0 = 2$

curves almost coincide, thus verifying our analysis. In Fig. 2, IOP is plotted versus SNR for non-identical Hoyt fading, after fixing C_0 to 2. For the 2×2 MIMO OSTBC system the channel specifications mentioned earlier were used, while for the 3×1 MIMO OSTBC system the parameters of the non-identical channels are $(\Omega_{1,1}, q_{1,1}) = (0.90, 0.49)$, $(\Omega_{2,1}, q_{2,1}) = (0.84, 0.89)$, $(\Omega_{3,1}, q_{3,1}) = (1.26, 0.92)$. Notice, that in all cases, the values of the Ω parameters are chosen such that the total power is equal to n_tn_r. The curves of Fig. 2 show that the saddlepoint approximation method achieves an excellent degree of accuracy in estimating the CDF, which results from the analysis of the original problem. Finally, in Fig. 3 the 5% theoretical outage capacity versus SNR is presented, i.e., the minimum capacity value that is supported with probability 95%.

V. Conclusion

In this paper, we deal with the problem of Hoyt fading in OSTBC MIMO systems. Useful results concerning the in-

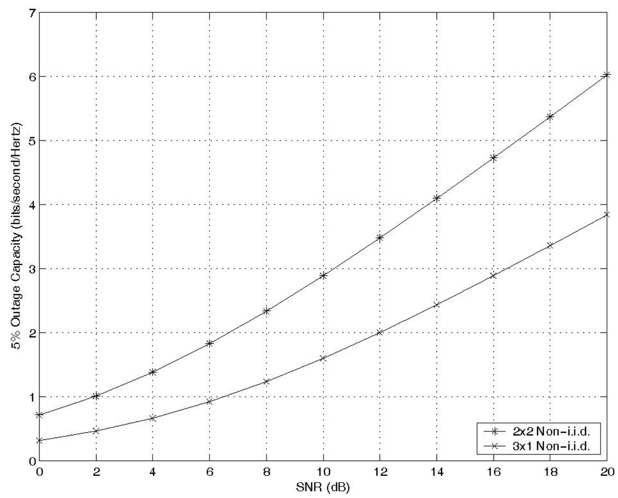

Fig. 3. 5% Outage Capacity vs SNR

formation outage probability for independent non-necessarily identical distributed Hoyt fading channels are presented. Specifically, exact closed-form expressions are derived for i.i.d. Hoyt fading. Moreover, the saddlepoint method is proposed to approximate the IOP for non-identical Hoyt fading channels. The method is very simple to implement and provides very accurate estimates of the IOP for various OTBC MIMO systems and operating SNRs. Potential application of the saddlepoint approximation method in performance evaluation of wireless communications systems is under current investigation.

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